Stationary response and singular limits in a linear zonal gyrokinetic model

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Abstract

We derive the stationary response of a linear periodic gyrokinetic model without assuming uniform kinetic relaxation. When the quadratic energy is coercive, the energy-orthogonal stationary projection is determined by the field operator $D_0 = C - R^*\Pi R$. Cesàro and Abel means converge strongly, but their operator-norm errors remain one when nonzero frequencies accumulate at zero. Adding the same constant drift detuning to every species removes the stationary space and generates a slow unitary evolution on its former kernel, so the long-time and zero-detuning limits do not commute. At vanishing polarization, a homogeneous kinetic-species model instead imposes charge and current constraints. We construct a self-adjoint evolution on the continuous-velocity constraint space and prove an $O(\sqrt{\varepsilon})$ error bound, in the energy coordinate, for one coercive regularization of constrained initial data; unrestricted data retain fast charge oscillations. In a specified 25-dimensional orbit discretization with trapped and passing particles, replacing the energy projection by ordinary orbit averaging changes the stationary field by 23.2%. The projection and singular-limit results are analytical; the percentage is stable under the reported grid refinements and is not a certified continuum value.

Keywords. Zonal gyrokinetics; orbit averaging; energy projection; coercive field equation; singular perturbation.

1 Introduction

A collisionless zonal perturbation may retain a stationary component after its oscillatory part has phase mixed. The residual zonal-flow problem asks how this component depends on the initial perturbation, magnetic geometry, and field constraint. The calculation of Rosenbluth and Hinton [9] and later finite-wavelength orbit formulas [7] show why this is a response problem rather than only an eigenvalue calculation.

The field changes which part of an initial kinetic perturbation is stationary. If Π is the orthogonal projection onto the kernel of uncoupled orbit transport, Πh_0 generally differs from the stationary component of the coupled evolution. Projection in the polarization-energy inner product instead gives the field equation

$$(C - R^*\Pi R)\phi_\infty = R^*\Pi M h_0.$$

The operator on the left is uniformly positive whenever the energy operator M is. It therefore admits a variational approximation and a residual estimate without a kinetic spectral gap. The dynamic field equation reduces to this stationary equation on the positive real Laplace axis.

Orbit averaging and decomposition into orbit invariants appear in residual-flow calculations [7, 8] and gyrokinetic eigenvalue methods [6]. Here we formulate the projection for a Hilbert-space field constraint, retain energy orthogonality, and separate the algebraic residual from quadrature error. Section 10 evaluates a periodic model with trapped and passing particles and directly compares the energy projection with ordinary orbit averaging.

Two limits govern the long-time behavior. Skew-adjointness in the energy norm gives strong Cesàro and Abel convergence, but frequencies accumulating at zero prevent operator-norm convergence. A bounded perturbation can nevertheless be averaged on the stationary space without a spectral gap, using the semigroup argument behind results such as [5]. For constant drift detuning, every nonzero detuning removes the stationary kernel, while the inverse-detuning time scale carries a nontrivial effective unitary evolution. This evolution is not decay of the kinetic norm.

The singular endpoint $M \geq 0$ requires a separate construction. In a homogeneous model with kinetic species, neutral data must also satisfy a current constraint. A finite-rank decomposition of velocity multiplication yields a self-adjoint generator on the continuous-velocity constraint space, with an explicit domain and dense core. A corrector then gives a $\sqrt{\varepsilon}$ error estimate for one regularization in which the energy operator becomes strictly positive. Strong coupling is known to produce projected dynamics [1, 3]; our calculation identifies the gyrokinetic constraint and its charge multiplier in an unbounded velocity space. Other small-wavenumber gyroaverage limits need not produce the same regularization.

Finally, we distinguish uncoupled orbit transport, the coupled generator, and an analytically continued scalar field form. Absolute continuity of the first does not imply the same spectral type for the second, and a zero of the continued form is not an off-axis eigenvalue of a unitary generator. This distinction determines the scope of the resonant compatibility criterion and the response formulas below.

2 The stationary projection as a field equation

Let $\mathcal{H}$ and $\mathcal{X}$ be complex Hilbert spaces. Let T_0 be skew-adjoint on $\mathcal{H}$, let $R : \mathcal{X} \to \mathcal{H}$ be bounded, and let $C = C^* \geq c_- \mathrm{Id} > 0$ be bounded on $\mathcal{X}$. Set

$$Q = C^{-1}R^*, \quad M = \mathrm{Id} - RC^{-1}R^*, \quad m_0 \mathrm{Id} \leq M \leq \mathrm{Id}, \quad A = M^{-1}T_0, \qquad \mathcal{D}(A) = \mathcal{D}(T_0). \tag{2.1}$$

The energy inner product is $\langle f, g \rangle_M = \langle f, Mg \rangle$. Let

$$N = \ker T_0, \qquad \Pi : \mathcal{H} \to N \tag{2.2}$$

be the orthogonal projection in the original inner product. The closed subspace N is contained in $\mathcal{D}(T_0)$.

Proposition 2.1 (Energy evolution)**.** *The operator A is skew-adjoint in the energy inner product, and $U(t) = e^{tA}$ is a strongly continuous unitary group there. Its kernel is N.*

Proof. Skew-symmetry follows from $\langle f, Ag \rangle_M = \langle f, T_0 g \rangle$. For $\lambda > 0$,

$$\lambda M - T_0 = [\mathrm{Id} - \lambda RC^{-1}R^*(\lambda - T_0)^{-1}](\lambda - T_0).$$

The factor subtracted from the identity has norm at most $1 - m_0$, so the range is all of $\mathcal{H}$. The same proof applies to $-T_0$. Thus A and $-A$ are maximal dissipative in the energy norm, which proves skew-adjointness and group generation [2]. Since M is invertible, $\ker A = \ker T_0$. □

Theorem 2.2 (Positive field equation)**.** *Let P_M be the energy-orthogonal projection onto N, and define*

$$D_0 = C - R^* \Pi R. \tag{2.3}$$

Then

$$m_0 C \le D_0 \le C, \tag{2.4}$$

so D_0 has a bounded inverse. For every $h \in \mathcal{H}$, the stationary field and state are

$$\phi_\infty := QP_M h = D_0^{-1} R^* \Pi M h, \tag{2.5}$$

$$P_M h = \Pi M h + \Pi R \phi_\infty, \tag{2.6}$$

$$P_M = \Pi - \Pi R D_0^{-1} R^* (\mathrm{Id} - \Pi). \tag{2.7}$$

Equivalently,

$$P_M = (\Pi M|_N)^{-1} \Pi M, \qquad \|(\Pi M|_N)^{-1}\| \le m_0^{-1}. \tag{2.8}$$

Proof. The assumption $RC^{-1}R^* \le (1-m_0)\mathrm{Id}$ implies $C^{-1/2}R^*RC^{-1/2} \le (1-m_0)\mathrm{Id}$ by equality of the norms of an operator and its adjoint. Thus $R^*\Pi R \le R^*R \le (1-m_0)C$, proving (2.4).

A vector k equals $P_M h$ precisely when $k \in N$ and $\Pi M(h-k) = 0$. On N, ΠM is self-adjoint and bounded below by m_0, which proves (2.8). Expanding M gives

$$k = \Pi M h + \Pi R Q k.$$

Apply R^* and use $CQk = R^*k$ to obtain $D_0 Qk = R^*\Pi M h$. This proves (2.5) and (2.6).

For the last formula, put $y = R^*(\mathrm{Id} - \Pi)h$ and define $k = \Pi h - \Pi R D_0^{-1} y$. It belongs to N, and

$$\Pi M(h-k) = \Pi R[-C^{-1} + (\mathrm{Id} - C^{-1}R^*\Pi R)D_0^{-1}]y = 0.$$

Hence $k = P_M h$, giving (2.7). □

The correction in (2.7) is not optional. It accounts for the fact that the field changes orthogonality while leaving the kernel unchanged. If $h \in N$, the correction vanishes and $P_M h = h$, as required. If $h \perp N$ in the original inner product, its stationary component need not vanish.

2.1 Variational approximation and residual bounds

Let $b = R^*\Pi M h$. Since D_0 is positive, ϕ_∞ uniquely minimizes

$$\mathcal{J}(\phi) = \tfrac{1}{2}\langle \phi, D_0 \phi\rangle - \Re\langle b, \phi\rangle. \tag{2.9}$$

Write $\|\phi\|_{D_0}^2 = \langle \phi, D_0\phi\rangle$ and $d_0 = m_0 c_-$.

Theorem 2.3 (Galerkin convergence and residual error)**.** *Let $X_n \subset \mathcal{X}$ be finite-dimensional, and let $\phi_n \in X_n$ satisfy*

$$\langle \psi, D_0 \phi_n\rangle = \langle \psi, b\rangle \quad \textit{for all } \psi \in X_n. \tag{2.10}$$

Then

$$\|\phi_\infty - \phi_n\|_{D_0} = \inf_{\psi \in X_n} \|\phi_\infty - \psi\|_{D_0}. \tag{2.11}$$

For any trial $\widetilde{\phi} \in \mathcal{X}$, with residual $r = b - D_0\widetilde{\phi}$,

$$\|\phi_\infty - \widetilde{\phi}\|_{D_0} \le \frac{\|r\|}{\sqrt{d_0}}, \tag{2.12}$$

$$\|\phi_\infty - \widetilde{\phi}\| \le \frac{\|r\|}{d_0}. \tag{2.13}$$

The stationary trial state $\widetilde{k} = \Pi M h + \Pi R \widetilde{\phi}$ satisfies

$$\|P_M h - \widetilde{k}\|_M \le \sqrt{\frac{1-m_0}{m_0}}\,\frac{\|r\|}{\sqrt{d_0}}, \tag{2.14}$$

$$Q\widetilde{k} - \widetilde{\phi} = C^{-1} r. \tag{2.15}$$

If X_n are nested with dense union, the Galerkin fields and stationary states converge for every fixed h.

Proof. The error in (2.10) is D_0-orthogonal to X_n, so the Pythagorean identity gives (2.11). For $e = \phi_\infty - \widetilde{\phi}$, $D_0 e = r$, and

$$\|e\|_{D_0}^2 = \langle e, r\rangle \le \|e\|\|r\| \le d_0^{-1/2}\|e\|_{D_0}\|r\|.$$

This proves (2.12) and (2.13). Moreover,

$$\|\Pi R e\|_M^2 \le \|\Pi R e\|^2 = \langle e, R^*\Pi R e\rangle \le \frac{1-m_0}{m_0}\langle e, D_0 e\rangle,$$

using (2.4). This is (2.14). Finally, $R^*\widetilde{k} = b + R^*\Pi R\widetilde{\phi} = C\widetilde{\phi} + r$, giving (2.15). Density and equivalence of the field norms imply convergence. □

The trial state lies in N and is stationary before the field equation is solved. Equation (2.15) measures whether the reconstructed field satisfies quasineutrality. To use the residual bound, errors in the assembled operators must also be controlled; a small residual for an inaccurate matrix is insufficient.

3 The dynamic field equation and its zero-frequency limit

The stationary equation is the zero-frequency limit of an invertible field equation on the positive real Laplace axis. This observation gives a way to approximate a damped time average without integrating the kinetic equation to a large final time. For $\Re p \neq 0$ set

$$Z(p) = (p - T_0)^{-1}, \qquad D(p) = C - pR^* Z(p) R. \tag{3.1}$$

No compactness assumption on R is made.

Theorem 3.1 (Resolvent reduction and uniform coercivity)**.** *For $\Re p \neq 0$, $D(p)$ is boundedly invertible and*

$$(p - A)^{-1} = \big[Z(p) + pZ(p)RD(p)^{-1}R^*Z(p)\big]M. \tag{3.2}$$

For every real $p > 0$,

$$\Re\langle\phi, D(p)\phi\rangle \ge m_0\langle\phi, C\phi\rangle, \qquad \|D(p)^{-1}\| \le (m_0 c_-)^{-1}. \tag{3.3}$$

Moreover $D(p) \to D_0$ and $D(p)^{-1} \to D_0^{-1}$ strongly as $p \downarrow 0$. The field of the Abel mean is the unique solution of

$$D(p)\phi_p = pR^*Z(p)Mh_0, \qquad \phi_p = pQ(p - A)^{-1}h_0, \tag{3.4}$$

and $\phi_p \to \phi_\infty$ in $\mathcal{X}$.

Proof. Both $p - T_0$ and $pM - T_0 = M(p - A)$ are bijective when $\Re p \neq 0$. Factor the second as

$$pM - T_0 = (p - T_0)[\mathrm{Id} - pZ(p)RC^{-1}R^*].$$

The elementary identity that $\mathrm{Id} - UV$ is invertible if and only if $\mathrm{Id} - VU$ is invertible shows that $\mathrm{Id} - pC^{-1}R^*Z(p)R = C^{-1}D(p)$ is invertible. Expanding the inverse of the factorization proves (3.2). Applying Q to the equation $(pM - T_0)\widehat{h} = Mh_0$ gives (3.4).

Let $H_0 = -\mathrm{i}T_0$. For $p > 0$, the spectral theorem gives

$$\Re\{p(p - T_0)^{-1}\} = p^2(p^2 + H_0^2)^{-1} \leq \mathrm{Id}.$$

Consequently $\Re D(p) \geq C - R^*R \geq m_0 C$. The same bound for the adjoint, or the already established invertibility, gives (3.3). The multipliers $p/(p - \mathrm{i}\lambda)$ converge pointwise to $\mathbf{1}_{\{0\}}(\lambda)$ and are bounded by one. Thus $pZ(p) \to \Pi$ strongly. It follows that $D(p) \to D_0$ strongly. For each $b \in \mathcal{X}$,

$$[D(p)^{-1} - D_0^{-1}]b = D(p)^{-1}[D_0 - D(p)]D_0^{-1}b \longrightarrow 0$$

by the uniform inverse bound. Taking the limit in (3.4) proves the last assertion. □

The bound in (3.3) is asserted on the positive real axis, not uniformly throughout a half-plane. Also, strong convergence of $D(p)$ does not imply convergence in operator norm.

Corollary 3.2 (A field error bound with assembly errors)**.** *Let D be either D_0 or $D(p)$ for real $p > 0$, and let $D\phi = b$. Suppose bounded approximations $\widetilde{D}, \widetilde{b}$ satisfy $\|D - \widetilde{D}\| \leq \delta_D$ and $\|b - \widetilde{b}\| \leq \delta_b$. For every trial field $\widetilde{\phi} \in \mathcal{X}$,*

$$\|\phi - \widetilde{\phi}\| \leq \frac{\|\widetilde{b} - \widetilde{D}\widetilde{\phi}\| + \delta_b + \delta_D\|\widetilde{\phi}\|}{m_0 c_-}. \tag{3.5}$$

Proof. The numerator bounds $\|b - D\widetilde{\phi}\|$ by the triangle inequality. Apply the inverse bound from Theorem 2.2 or Theorem 3.1. □

An algebraic residual for an assembled matrix is only the first term in this estimate. A difference between two quadrature grids does not by itself certify δ_D or δ_b.

4 A periodic linear gyrokinetic model

4.1 Transport and field energy

Let θ lie on a circle of length ℓ, and let $B, \beta > 0$ be smooth periodic functions. For finitely many species define

$$E_s = \tfrac{1}{2}m_s v^2 + \mu B(\theta), \qquad F_s(E) = \left(\frac{m_s}{2\pi}\right)^{3/2} e^{-E}, \qquad \mathrm{d}^3 v_s = \frac{2\pi B}{m_s}\,\mathrm{d}v\,\mathrm{d}\mu, \tag{4.1}$$

$$\sigma = (B\beta)^{-1}, \qquad \mathcal{H} = \bigoplus_s L^2(\sigma\,\mathrm{d}\theta\,\mathrm{d}^3 v_s/F_s), \qquad \mathcal{X} = L^2(\sigma\,\mathrm{d}\theta). \tag{4.2}$$

Take real periodic J_s with $|J_s| \leq 1$, and set

$$(R\phi)_s = q_s F_s J_s \phi, \qquad R^*h = \sum_s q_s \int J_s h_s\,\mathrm{d}^3 v_s. \tag{4.3}$$

The characteristic flow is generated by

$$X_s = v\beta\partial_\theta - \frac{\mu}{m_s}\beta B'\partial_v, \qquad T_0 = -X - \mathrm{i}\Omega, \tag{4.4}$$

where Ω_s is real and locally smooth. Let $U_0(t)$ be the complete characteristic group obtained from this flow and the unit-modulus phase generated by Ω_s, and define T_0 as its generator on the maximal generator domain. The group preserves E_s, μ, and $\sigma\,\mathrm{d}\theta\,\mathrm{d}^3v_s/F_s$, so it is unitary and T_0 is skew-adjoint. The collisionless zonal equation has no equilibrium-gradient drive in this model:

$$M\partial_t h = T_0 h, \qquad M = \mathrm{Id} - RC^{-1}R^*. \tag{4.5}$$

The operator statements below apply whenever $M \geq m_0\mathrm{Id} > 0$.

The electrostatic field term in the energy is

$$\|h\|_M^2 = \|h\|^2 - \|C^{-1/2}R^*h\|^2. \tag{4.6}$$

This is the quadratic form conserved by (4.5). It should not be replaced by $\|Mh\|^2$; the two generally differ. Positive gyrokinetic quadratic energies are also used to bound instability growth in driven systems [4].

Proposition 4.1 (Two positive zonal constraints)**.** *Assume singly charged species at equal normalized temperature and density, and let* $\Gamma_s(\theta) = \int F_s J_s^2\,\mathrm{d}^3v_s$. *For one kinetic ion species with modified-adiabatic electrons, set*

$$C = 2\mathrm{Id} - P_{\mathrm{av}}, \qquad P_{\mathrm{av}}\phi = \frac{\int \sigma\phi\,\mathrm{d}\theta}{\int \sigma\,\mathrm{d}\theta}. \tag{4.7}$$

If $\Gamma_{i,\max} < 1$, *then* $M \geq (1 - \Gamma_{i,\max})\mathrm{Id}$. *For kinetic ions and electrons,* $C = 2\mathrm{Id}$ *and*

$$M \geq \left[1 - \tfrac{1}{2}\sup_\theta(\Gamma_i + \Gamma_e)\right]\mathrm{Id}. \tag{4.8}$$

Proof. The operator R^*R is multiplication by $\sum_s q_s^2\Gamma_s$. In (4.7), $C^{-1} = (\mathrm{Id} + P_{\mathrm{av}})/2 \leq \mathrm{Id}$, so $\|RC^{-1}R^*\| \leq \|R\|^2 = \Gamma_{i,\max}$. For kinetic electrons, the norm is at most half the displayed sum. Since $RC^{-1}R^* \geq 0$, both lower bounds follow. □

For the usual Bessel gyroaverage at a fixed nonzero radial wavenumber, $\Gamma_s < 1$, and compactness of the circle gives the required uniform bound when the coefficients vary continuously. The averaging measure in (4.7) must agree with the Hilbert-space adjoint in (4.3). A different imposed field constraint requires its own self-adjointness and positivity check.

4.2 The orbit projection

On a regular periodic characteristic, let τ be travel time and $T(E, \mu)$ its period. Passing trajectories have two orientation labels; trapped trajectories carry their well label. The orbit measure is

$$\mathrm{d}m_s = \frac{2\pi}{m_s^2}\,\mathrm{d}\tau\,\mathrm{d}E\,\mathrm{d}\mu. \tag{4.9}$$

Separatrices and stationary phase-space points may be omitted when they have measure zero, as in the single-well geometry used below.

Proposition 4.2 (Phase-corrected orbit average)**.** *Suppose that the drift has zero time average on almost every regular periodic characteristic:*

$$\int_0^{T(E,\mu)} \Omega_s(\tau, E, \mu)\, \mathrm{d}\tau = 0. \tag{4.10}$$

Set $\Psi_s(\tau) = \int_0^\tau \Omega_s(r)\, \mathrm{d}r$*. Then* $N = \ker T_0$ *consists of the functions*

$$h_s(\tau, E, \mu) = e^{-\mathrm{i}\Psi_s(\tau,E,\mu)} a_s(E, \mu) \tag{4.11}$$

on each orbit family, with arbitrary square-integrable amplitudes. Its orthogonal projection is

$$(\Pi h)_s(\tau, E, \mu) = e^{-\mathrm{i}\Psi_s(\tau,E,\mu)} \frac{1}{T} \int_0^T e^{\mathrm{i}\Psi_s(r,E,\mu)} h_s(r, E, \mu)\, \mathrm{d}r. \tag{4.12}$$

In particular, N is infinite-dimensional whenever a regular orbit family has positive invariant measure.

Proof. By (4.10), multiplication by $e^{-\mathrm{i}\Psi_s}$ is a periodic unitary gauge. Direct differentiation gives

$$(-\partial_\tau - \mathrm{i}\Omega_s)(e^{-\mathrm{i}\Psi_s} a) = e^{-\mathrm{i}\Psi_s}(-\partial_\tau a).$$

The kernel of the derivative on a circle consists of constants, and its orthogonal projection is the ordinary time average. The weight $F_s(E)^{-1}$ is constant on each orbit, so it does not change that projection. Integrating the fiberwise formula with (4.9) proves the result on $\mathcal{H}$. The kernel and domain statements also follow from the direct-integral characteristic group, not from a formal boundary calculation alone. □

Corollary 4.3 (A geometric sufficient condition)**.** *On $(-\pi, \pi]$, suppose B and β are even, B is strictly increasing on $(0, \pi)$ with a single nondegenerate minimum and maximum, and*

$$\Omega_s(-\theta, v, \mu) = -\Omega_s(\theta, v, \mu), \qquad \Omega_s(\theta, -v, \mu) = \Omega_s(\theta, v, \mu). \tag{4.13}$$

Then (4.10) *holds for both passing orientations and for trapped trajectories. The formulas of Theorems 2.2 and 2.3 are therefore explicit orbit-integral formulas for this gyrokinetic class.*

Proof. At fixed E, μ, the positive speed $|v(\theta)|$ is even. Hence $\Omega_s/(\beta|v|)$ is odd. Its integral over a passing period vanishes. The trapped interval is symmetric about zero; each half of the bounce has the same even time weight, and (4.13) gives zero there as well. The nondegenerate extrema ensure that exceptional separatrices form a null set; their unbounded periods do not alter the almost-everywhere argument. □

The constancy of β and a particular circular equilibrium formula are not needed. Conversely, (4.12) is not asserted solely from periodicity: a nonzero orbit-averaged drift changes the fiber kernel. The abstract field equation remains valid with the actual projection onto that kernel.

5 Orbit holonomy and resonant compatibility

The zero-mean condition (4.10) is sufficient for a stationary component on every regular orbit. Without it, the existence of such a component is decided separately on each orbit.

Proposition 5.1 (The kernel without a symmetry assumption)**.** *Let* $\mathfrak{h}_s(E,\mu) = \int_0^T \Omega_s \,\mathrm{d}\tau$ *on each regular periodic orbit. Its contribution to* Π *is zero unless* $\mathfrak{h}_s(E,\mu) \in 2\pi\mathbb{Z}$*. On the resonant set the projection is exactly* (4.12)*, with* $\Psi_s(\tau) = \int_0^\tau \Omega_s \,\mathrm{d}r$*. Thus a resonant curve of zero invariant measure does not by itself produce a nonzero stationary state in* $\mathcal{H}$*.*

Proof. The equation $T_0 h = 0$ on an orbit gives $h(\tau) = e^{-\mathrm{i}\Psi_s(\tau)} h(0)$. Periodicity permits a nonzero value of $h(0)$ exactly when $e^{-\mathrm{i}\mathfrak{h}_s} = 1$. On that set the displayed phase is periodic and has unit modulus, so orthogonal projection gives (4.12). Direct integration of the fibers proves the assertion, including the null-set statement. □

5.1 The spectral measure of uncoupled orbit transport

Assume now (4.10). Suppose almost every characteristic belongs to a regular passing or trapped family, with pitch $\lambda = \mu/E$ and period

$$T_s(E,\lambda E) = E^{-1/2}\mathcal{T}_s(\lambda), \qquad E > 0. \tag{5.1}$$

The function $\mathcal{T}_s$ is finite and positive on each open regular pitch interval; divergence at its endpoints is allowed. This includes smooth single-well geometries with nondegenerate extrema. Assume also that the orbit labels and travel-time coordinates give a measurable direct-integral decomposition of $\mathcal{H}$ outside the separatrix set, which has measure zero. An orientation or well label is understood in all sums below. Define

$$e_{s,n}(\tau;E,\lambda) = e^{-\mathrm{i}\Psi_s(\tau)} e^{-2\pi \mathrm{i} n\tau/T_s}, \qquad \omega_{s,n}(E,\lambda) = \frac{2\pi n\sqrt{E}}{\mathcal{T}_s(\lambda)}, \tag{5.2}$$

$$c_{s,n}[\phi](E,\lambda) = \frac{1}{T_s}\int_0^{T_s} \overline{e_{s,n}(\tau)}\, q_s F_s(E) J_s(\tau)\phi(\theta(\tau))\,\mathrm{d}\tau, \tag{5.3}$$

$$\mathrm{d}\nu_s = \frac{2\pi E T_s(E,\lambda E)}{m_s^2 F_s(E)}\,\mathrm{d}E\,\mathrm{d}\lambda. \tag{5.4}$$

The factor E in (5.4) is the Jacobian of $\mu = E\lambda$.

Theorem 5.2 (An atom and an absolutely continuous transport measure)**.** *Under these assumptions,* $H_0 = -\mathrm{i}T_0$ *is unitarily equivalent to multiplication by* $\omega_{s,n}$ *on the direct sum of the spaces* $L^2(\,\mathrm{d}\nu_s)$*. Its only point spectrum is zero, its zero eigenspace is* N*, and its restriction to* $N^\perp$ *is purely absolutely continuous. For fields* $\psi, \phi \in \mathcal{X}$*, the matrix element of its spectral measure between* $R\psi$ *and* $R\phi$ *has a zero atom of mass* $\langle \psi, R^*\Pi R\phi\rangle$*. Away from zero it has the* L^1 *density*

$$\rho_{\psi,\phi}(\omega) = \sum_{s,n:\, n\omega>0} \int \frac{2\pi}{m_s^2}\frac{E_n T_s(E_n,\lambda E_n)}{F_s(E_n)}\frac{2E_n}{|\omega|}\,\overline{c_{s,n}[\psi](E_n,\lambda)}\, c_{s,n}[\phi](E_n,\lambda)\,\mathrm{d}\lambda, \tag{5.5}$$

$$E_n = \left(\frac{\omega\mathcal{T}_s(\lambda)}{2\pi n}\right)^2.$$

The density is defined almost everywhere; no pointwise trace estimate for arbitrary fields is asserted.

Proof. The periodic gauge and Fourier series on each orbit diagonalize T_0 with the frequencies in (5.2). Parseval's identity and (4.9) give the weight (5.4). For $n = 0$ the multiplier is zero. For each $n \neq 0$ and fixed regular λ, the map $E \mapsto \omega_{s,n}$ is continuously differentiable and strictly monotone, with nonzero derivative on $E > 0$. It therefore pushes an absolutely continuous measure

in E to an absolutely continuous measure in ω. Tonelli's theorem applied to squared coefficients proves the spectral assertion. The change of variables has Jacobian $|\,\mathrm{d}E/\,\mathrm{d}\omega| = 2E_n/|\omega|$ and gives (5.5). Absolute integrability for diagonal elements follows from $\|R\phi\|^2 < \infty$; Cauchy–Schwarz or polarization gives the off-diagonal statement. □

This theorem describes T_0. The field term changes the coupled generator to $A = M^{-1}T_0$; its spectral type cannot be read off from the uncoupled direct integral. Orbit-invariant discretizations likewise couple their orbit blocks through the field equation [6].

5.2 What an embedded eigenfunction must satisfy

For a nonzero real ω define the quotient condition

$$\sum_{s,n} \int \frac{|c_{s,n}[\phi](E,\lambda)|^2}{|\omega_{s,n}(E,\lambda) - \omega|^2}\,\mathrm{d}\nu_s < \infty. \tag{5.6}$$

It is a condition on the field, not an automatic consequence of $\phi \in \mathcal{X}$.

Theorem 5.3 (Exact criterion for a nonzero embedded eigenvalue)**.** *Under Theorem 5.2,* $\mathrm{i}\omega$*,* $\omega \neq 0$*, is an eigenvalue of* A *if and only if there exists* $0 \neq \phi \in \mathcal{X}$ *satisfying* (5.6) *and the following self-consistency equation. Form the unique* $h \in \mathcal{H}$ *with orbit Fourier coefficients*

$$h_{s,n} = -\frac{\omega\, c_{s,n}[\phi]}{\omega_{s,n} - \omega}. \tag{5.7}$$

Then the equation is $C\phi = R^*h$*. When it holds,* $h \in \mathcal{D}(A)$ *and* $Ah = \mathrm{i}\omega h$*.*

Proof. An eigenfunction satisfies $(H_0 - \omega)h = -\omega R\phi$, where $\phi = Qh$. Theorem 5.2 gives $\ker(H_0 - \omega) = 0$, so ϕ cannot be zero and the coefficients must be (5.7). Their square integrability is precisely (5.6). Conversely, that condition defines $h \in \mathcal{H}$. The identity $\omega_{s,n}h_{s,n} = \omega h_{s,n} - \omega c_{s,n}[\phi]$ shows $H_0 h \in \mathcal{H}$, hence $h \in \mathcal{D}(H_0)$. Self-consistency then gives $H_0 h = \omega M h$, which is the eigenvalue equation. □

If the coefficients are continuous in E near a regular resonance for almost every λ, condition (5.6) forces

$$c_{s,n}[\phi](E_n, \lambda) = 0 \tag{5.8}$$

for almost every such pitch. Indeed, a nonzero continuous value makes the local integral of $(E - E_n)^{-2}|c_{s,n}|^2$ diverge. Vanishing with a uniform local Lipschitz bound is sufficient for local integrability on a compact regular resonance strip, but global integrability over all harmonics and pitch endpoints is still required. Equation (5.8) alone therefore neither excludes nor constructs an embedded eigenvalue. No limiting-absorption principle or meromorphic continuation is assumed in these results.

Corollary 5.4 (The zero-drift, zero-gyroaverage moment problem)**.** *For* $\Omega_s = 0$ *and* $J_s = 1$*, suppose both orientations of a regular passing family are present. The resonant moment conditions* (5.8)*, for one fixed* $\omega \neq 0$*, force* ϕ *to be spatially constant. Conversely, a constant field satisfies those conditions.*

Proof. After division by $q_s F_s(E)$, the coefficient on a passing orbit is a Fourier coefficient of $\phi(\theta(\eta,\lambda))$ independent of E. The resonance conditions therefore set every nonzero harmonic with the sign of ω to zero. Reversing the orientation reverses the harmonic index, so the opposite harmonics vanish as well. Outside a single null set of pitches, all nonzero Fourier coefficients vanish. For any such regular pitch, travel time is a smooth bijective parametrization of the spatial circle, and the field is constant almost everywhere. A constant has no nonzero orbit Fourier coefficients, which proves the converse. □

This is a moment statement for orbit transport. At a singular kinetic-species field constraint it does not replace the constrained evolution constructed in Section 9.

5.3 Scalar continuation under an explicit analytic bound

The orbit density permits a scalar continuation under additional hypotheses that control the whole pitch range. This is stronger than the mere existence of an L^1 density and weaker than continuation in the operator norm of $\mathcal{X}$.

Use $\eta = \tau/T_s \in [0,1]$ on every regular orbit. Suppose that there are finitely many orbit families and that, in each one,

$$\Psi_s(\eta;E,\lambda) = \sqrt{E}\,\psi_s(\eta,\lambda), \qquad J_s(\eta;E,\lambda) = J_0(\sqrt{E}\,j_s(\eta,\lambda)), \tag{5.9}$$

where ψ_s, j_s are real and uniformly bounded. Assume also

$$\sum_{s,\text{families}} \int \mathcal{T}_s(\lambda)^4 \,\mathrm{d}\lambda < \infty. \tag{5.10}$$

Logarithmic separatrix divergence is allowed by this condition. The spatial map $\theta(\eta,\lambda)$ is independent of E, as in the preceding orbit decomposition.

Proposition 5.5 (Continuation of bounded-field matrix elements)**.** *For bounded fields ψ, ϕ, the two restrictions of $\rho_{\psi,\phi}$ to the positive and negative real axes have holomorphic extensions to the respective components of*

$$\mathcal{S} = \{z \in \mathbb{C} : |\Im z| < |\Re z|\}.$$

The matrix element $\langle \psi, D(p)\phi\rangle$ from $\Re p > 0$ therefore continues across every nonzero point of the imaginary axis into $\{p : |\Re p| < |\Im p|\}$. On its left-half-plane part the continuation is

$$\langle \psi, D^{\mathrm{cont}}(p)\phi\rangle = \langle \psi, D(p)\phi\rangle - 2\pi p\, \rho_{\psi,\phi}(-\mathrm{i}p). \tag{5.11}$$

The notation on the left denotes a scalar form; no bounded operator $D^{\mathrm{cont}}(p)$ on $\mathcal{X}$ is asserted.

Proof. Define the entire function

$$a_{s,n}[\phi](z,\lambda) = \int_0^1 e^{\mathrm{i}z\psi_s(\eta,\lambda)} J_0(z j_s(\eta,\lambda))\phi(\theta(\eta,\lambda)) e^{2\pi \mathrm{i} n\eta}\,\mathrm{d}\eta.$$

The integral representation of J_0 gives $|J_0(zj)| \le e^{|j\Im z|}$. Hence $|a_{s,n}[\phi](z,\lambda)| \le \|\phi\|_\infty e^{K|\Im z|}$, uniformly in pitch and harmonic. Write $a^{\#}[\psi](z) = \overline{a[\psi](\overline{z})}$, which is also entire. Since $c_{s,n}[\phi] = q_s F_s(E) a_{s,n}[\phi](\sqrt{E},\lambda)$, substitution in (5.5) expresses each summand on a real half-axis as

$$\frac{q_s^2 F_s(0)}{2\pi^2 m_s^2} \frac{z^2 \mathcal{T}_s(\lambda)^4}{|n|^3} e^{-z_n^2} a^{\#}_{s,n}[\psi](z_n,\lambda) a_{s,n}[\phi](z_n,\lambda), \qquad z_n = \frac{z\mathcal{T}_s(\lambda)}{2\pi n},$$

with $n > 0$ on the positive component and $n < 0$ on the negative component. On a compact subset of either component, $\Re(z^2) \geq c > 0$. The absolute value is bounded by a constant times

$$\|\psi\|_\infty \|\phi\|_\infty \frac{\mathcal{T}_s^4}{|n|^3} \exp\left(-\frac{c\mathcal{T}_s^2}{4\pi^2 n^2} + K' \frac{\mathcal{T}_s}{|n|} \right).$$

The exponential is uniformly bounded by completing the square. Condition (5.10) and $\sum_{n\neq 0} |n|^{-3} < \infty$ give locally uniform absolute convergence. The resulting holomorphic functions agree almost everywhere with the density, and hence choose analytic representatives on the two real half-axes.

The spectral theorem writes the dynamic form as a constant minus $p \int_{\mathbb{R}} \rho_{\psi,\phi}(x)/(p - \mathrm{i}x)\,\mathrm{d}x$, with its zero atom absorbed into the constant. Near a nonzero axis point split the integral into a small interval and its complement. The complement is holomorphic there. Deforming the contour in the small interval, or evaluating its simple Cauchy jump, gives (5.11). This proves the continuation without any claim about the origin or the L^2 operator norm. □

The explicit model in Section 10 satisfies these hypotheses: one may take $\Psi = -\alpha v$, so $|\Psi|/\sqrt{E} \leq \alpha\sqrt{2}$, and the Bessel argument divided by $\sqrt{E}$ is bounded because $B \geq 1$ and $0 < \lambda < 1$. Its period functions have only logarithmic divergence. A continued zero is not exhibited here. Failure of the displayed series outside the sectors would not prove failure of every possible continuation.

6 Averaged limits without uniform relaxation

Theorem 6.1 (Strong means and their exact norm error)**.** *For the abstract model* (2.1)*,*

$$\frac{1}{T}\int_0^T U(t)h\,\mathrm{d}t \longrightarrow P_M h \qquad (T \to \infty), \tag{6.1}$$

$$p(p - A)^{-1}h \longrightarrow P_M h \qquad (p \downarrow 0), \tag{6.2}$$

strongly in the energy norm, for every $h \in \mathcal{H}$. If nonzero points of $\sigma(A)$ accumulate at zero, then for every $T, p > 0$,

$$\left\| \frac{1}{T}\int_0^T U(t)\,\mathrm{d}t - P_M \right\|_M = 1, \tag{6.3}$$

$$\|p(p - A)^{-1} - P_M\|_M = 1. \tag{6.4}$$

Proof. In the energy Hilbert space, $H = -\mathrm{i}A$ is self-adjoint. The spectral multipliers for the two means are

$$f_T(\lambda) = \begin{cases} (e^{\mathrm{i}\lambda T} - 1)/(\mathrm{i}\lambda T), & \lambda \neq 0, \\ 1, & \lambda = 0, \end{cases} \qquad g_p(\lambda) = \frac{p}{p - \mathrm{i}\lambda}.$$

Their absolute values are at most one, and they converge pointwise to the indicator of $\{0\}$. The spectral theorem and dominated convergence give (6.1)–(6.2). On the energy-orthogonal complement of N, their norms are at most one. Nonzero spectral values tending to zero make $|f_T|$ and $|g_p|$ tend to one for each fixed T, p, proving both lower bounds. □

The spectral hypothesis holds for the kinetic class above. A short local construction makes this independent of the orbit decomposition or symmetry.

Proposition 6.2 (The periodic spectrum)**.** *Under* (4.1)–(4.5)*, with* $M \geq m_0 \mathrm{Id} > 0$*, one has* $\sigma(A) = \mathrm{i}\mathbb{R}$*. Every point of the axis has a normalized weakly null approximate eigenvector. Consequently* (6.3) *and* (6.4) *hold for this model.*

Proof. Choose a proper open subinterval of the circle and $0 \neq \chi \in C_c^\infty$ in that interval. For one species let

$$\mu = cr, \quad E = c(q + B_* r), \quad v = \sqrt{c}\, w(\theta, q, r), \quad w = \sqrt{2\{q + r(B_* - B)\}/m_s},$$

where (q, r) lies in a fixed positive rectangle and $B_* = \max B$. Put

$$N(q, r) = \int \frac{|\chi|^2}{\beta w}\, \mathrm{d}\theta, \qquad \Phi_{\omega,c}(\theta) = \int_{\theta_0}^{\theta} \frac{\omega + \Omega_s}{\beta v}\, \mathrm{d}s.$$

For a unit smooth amplitude $g(q, r)$ define, on the positive branch,

$$u_c = \frac{m_s}{\sqrt{2\pi}} c^{-3/4} F_s(E)^{1/2} N^{-1/2} \chi g e^{-\mathrm{i}\Phi_{\omega,c}},$$

and set it to zero elsewhere. Formula (4.9), or the equivalent change of variables in θ, gives $\|u_c\| = 1$. All factors except the envelope and phase are orbit invariants, and $X\Phi_{\omega,c} = \omega + \Omega_s$. It follows that

$$\|(T_0 - \mathrm{i}\omega)u_c\| = O(\sqrt{c}).$$

Its velocity support has volume $O(c^{3/2})$ at each θ, so weighted Cauchy–Schwarz gives $\|Qu_c\| = O(c^{3/4})$. Therefore

$$(A - \mathrm{i}\omega)u_c = M^{-1}[(T_0 - \mathrm{i}\omega)u_c + \mathrm{i}\omega RQu_c] \longrightarrow 0.$$

The shrinking support $E \leq Kc$ makes the sequence weakly null. Also $\|u_c\|_M \to 1$, because $\langle u_c, (\mathrm{Id} - M)u_c\rangle = \langle R^* u_c, C^{-1} R^* u_c\rangle \to 0$. Thus every $\mathrm{i}\omega$ lies in the spectrum. Skew-adjointness in the energy norm gives the reverse inclusion. □

For $h \notin N$, the unaveraged full state cannot converge strongly to $P_M h$, since

$$\|U(t)h - P_M h\|_M = \|h - P_M h\|_M. \tag{6.5}$$

Weak convergence on the complement, and ordinary convergence of the field, depend on additional spectral or mixing information. Neither follows merely from the existence of the strong averaged limits. In particular, the analysis does not assume the absence of embedded eigenvalues or singular continuous spectrum.

7 A bounded weak-relaxation limit

Let D be bounded, self-adjoint, and nonnegative in the energy Hilbert space. Consider

$$\partial_t h_\varepsilon = (A - \varepsilon D)h_\varepsilon, \qquad \varepsilon > 0. \tag{7.1}$$

It generates a contraction semigroup. On the time scale $s = \varepsilon t$, the generator is $A/\varepsilon - D$. Define the bounded nonnegative operator on N

$$D_N = P_M D|_N. \tag{7.2}$$

The projection here is the energy projection, not Π.

Theorem 7.1 (Averaging for initially stationary data)**.** *For every $h \in N$ and $S < \infty$,*

$$\sup_{0\le s\le S} \left\| e^{s(A/\varepsilon - D)}h - e^{-sD_N}h \right\|_M \longrightarrow 0 \qquad (\varepsilon \downarrow 0). \tag{7.3}$$

No spectral gap or quantitative mixing estimate is required.

Proof. Write $U_\varepsilon(t) = U(t/\varepsilon)$. The strong mean convergence in (6.1) implies, for every fixed $x \in \mathcal{H}$,

$$\sup_{0\le s\le S} \left\| \int_0^s U_\varepsilon(r)(\mathrm{Id} - P_M)x \,\mathrm{d}r \right\|_M \to 0. \tag{7.4}$$

For completeness, choose T_0 so that the norm of the mean over any interval $[0, T]$, $T \ge T_0$, is at most $\delta\|x\|_M$. In (7.4), the part with $s/\varepsilon \ge T_0$ is then at most $S\delta\|x\|_M$, while the remaining part is at most $\varepsilon T_0\|x\|_M$.

Step-function approximation and contractivity now show that every continuous $f : [0, S] \to \mathcal{H}$ satisfies

$$\int_0^s U_\varepsilon(s - r)f(r)\,\mathrm{d}r \longrightarrow \int_0^s P_M f(r)\,\mathrm{d}r \tag{7.5}$$

strongly and uniformly for $s \in [0, S]$. For a step function, each integral over an interval is a difference of primitives in (7.4); uniform approximation then handles general f.

Use the norm-convergent Dyson expansion

$$e^{s(A/\varepsilon - D)}h = \sum_{j=0}^{\infty} y_{j,\varepsilon}(s), \qquad y_{0,\varepsilon}(s) = h, \qquad y_{j+1,\varepsilon}(s) = -\int_0^s U_\varepsilon(s - r)Dy_{j,\varepsilon}(r)\,\mathrm{d}r.$$

The initial equality uses $h \in N$. Induction with (7.5) gives

$$y_{j,\varepsilon}(s) \longrightarrow \frac{(-s)^j}{j!} D_N^j h$$

uniformly on $[0, S]$. The bound $\|y_{j,\varepsilon}(s)\|_M \le S^j \|D\|_M^j \|h\|_M / j!$ permits passage through the series and proves (7.3). □

Corollary 7.2 (The field on the slow time scale)**.** *For the same data,*

$$Qe^{s(A/\varepsilon - D)}h \longrightarrow Qe^{-sD_N}h \tag{7.6}$$

uniformly for s in compact intervals. If $h = P_M h_0$, Theorem 2.2 supplies this initial stationary state and its field without a long collisionless simulation.

Proof. The map Q is bounded, so (7.6) follows directly from (7.3). □

The stationary space remains infinite-dimensional. Equation (7.3) is a reduction to that space, not automatically to finitely many moments. Finite-dimensional approximation of D_N requires a specified approximation space and data regularity. Moreover, a physical velocity-differential collision operator is generally unbounded in the energy space. Applying this result to such an operator would require domain estimates that are not supplied by boundedness of a finite discretization.

7.1 Why operator-norm convergence fails in general

Proposition 7.3 (A sharp limitation of averaging)**.** *There exist a skew-adjoint A, a bounded nonnegative self-adjoint D, and an infinite-dimensional $N = \ker A$ for which* (7.3) *holds for each $h \in N$, but, for every fixed $s > 0$,*

$$\left\| e^{s(A/\varepsilon - D)}|_N - e^{-sD_N} \right\| \not\longrightarrow 0. \tag{7.7}$$

Proof. Take $\mathcal{H} = \bigoplus_{n\geq 1} \mathbb{C}^2$ with its usual norm, and define blockwise

$$A_n = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i}/n \end{pmatrix}, \qquad D_n = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{7.8}$$

Here A is skew-adjoint and $0 \leq D \leq 2\mathrm{Id}$. The kernel is the sum of the first coordinate in each block, and $D_N = \mathrm{Id}_N$. Let h_n be the first unit coordinate in block n and set $\varepsilon_n = n^{-1/2}$. On that block, $A_n/\varepsilon_n \to 0$, so

$$e^{s(A/\varepsilon_n - D)} h_n \longrightarrow \begin{pmatrix} (1+e^{-2s})/2 \\ (1-e^{-2s})/2 \end{pmatrix}$$

in the two block coordinates. The effective state is $e^{-s}h_n$, whose second coordinate is zero. The norm difference therefore has lower limit at least $(1-e^{-2s})/2 > 0$. There is no conflict with (7.3): the initial vector changes with n. □

The example is an abstract limitation of the theorem, not a discretization of a gyrokinetic collision operator. It isolates the role of frequencies that approach zero faster than the relaxation scale.

8 Small drift detuning and the order of limits

The stationary projection can change discontinuously even when the generator changes by a bounded perturbation of arbitrarily small norm. The orbit holonomy gives an explicit example. This also identifies a slow evolution on the old stationary space.

Lemma 8.1 (Averaging a general bounded perturbation)**.** *If B is any bounded operator on the energy Hilbert space, then for every $h \in N$,*

$$e^{s(A/\varepsilon + B)} h \longrightarrow e^{sB_N} h, \qquad B_N = P_M B|_N, \tag{8.1}$$

strongly and uniformly on compact intervals of $s \geq 0$.

Proof. In the proof of Theorem 7.1, replace $-D$ by B. The convolution convergence (7.5) is unchanged. Its jth Dyson term converges to $s^j B_N^j h/j!$ and is bounded by $S^j \|B\|_M^j \|h\|_M / j!$. This summable majorant proves the assertion. Positivity and self-adjointness were needed for contraction in the relaxation application, not for this bounded-perturbation argument. □

Theorem 8.2 (Loss of the stationary space under constant detuning)**.** *Assume the regular orbit hypotheses of Theorem 5.2, including zero mean drift. Add a constant $\delta \neq 0$ to every species' drift and leave the field constraint unchanged. The coupled generator becomes*

$$A_\delta = A - \mathrm{i}\delta M^{-1}. \tag{8.2}$$

Then $\ker A_\delta = \{0\}$ *for every* $\delta \neq 0$. *Its Cesàro and Abel means converge strongly to zero, whereas the corresponding limits at* $\delta = 0$ *equal* P_M. *In particular, the limits* $T \to \infty$ *and* $\delta \to 0$ *of the Cesàro mean do not commute whenever* $P_M h \neq 0$.

For initially stationary data $h \in N$ *and* $\delta \downarrow 0$,

$$e^{(s/\delta)A_\delta} h \longrightarrow e^{-\mathrm{i}sB_N} h, \qquad B_N = (\Pi M|_N)^{-1}, \tag{8.3}$$

strongly and uniformly for s *in compact intervals. The effective operator* B_N *is positive and self-adjoint in the restricted energy inner product, with spectrum contained in* $[1, m_0^{-1}]$.

Proof. The new holonomy is $\delta T_s(E, \lambda E)$. By Proposition 5.1, a kernel fiber requires $\delta \mathcal{T}_s(\lambda)/\sqrt{E} = 2\pi n$. The integer $n = 0$ gives no solution. For each $n \neq 0$ this equation defines at most one positive energy at each regular pitch. Fubini's theorem shows that the union of these graphs has zero invariant measure. Thus the kernel of $T_0 - \mathrm{i}\delta$ is zero, and invertibility of M proves the kernel statement for A_δ. The perturbed transport is skew-adjoint and gives the same unitary energy construction. The spectral theorem proves convergence of its means to zero.

For each fixed T, the bounded perturbation formula gives strong convergence of the means to those of A as $\delta \to 0$. Their subsequent limit is P_M; reversing the two limits gives zero. Apply Lemma 8.1 with $B = -\mathrm{i}M^{-1}$ and $\varepsilon = \delta$ to obtain (8.3) with $B_N = P_M M^{-1}|_N$. Formula (2.8) shows that this compression is $(\Pi M|_N)^{-1}$. Since $m_0 \mathrm{Id}_N \leq \Pi M|_N \leq \mathrm{Id}_N$, its inverse has the stated positivity, self-adjointness, and spectral bounds. □

A long-lived intermediate zonal response is already part of gyrokinetic residual-flow theory [8]. The identical additive detuning in Theorem 8.2 is a solvable perturbation that demonstrates noncommuting limits on its inverse scale; it is not a generic model of departures from an optimized magnetic geometry. The disappearance of the stationary projection also does not imply strong decay of a unitary kinetic state.

9 A constrained continuum limit at vanishing polarization

The inverse M^{-1} ceases to exist when the field response has a null direction. In a homogeneous kinetic-species model this singularity can be resolved exactly. The constraint has to be imposed before the transport operator is inverted. The result below addresses this model in the continuous velocity space, not only after a finite moment truncation.

9.1 The modified-adiabatic gauge at zero wavenumber

For one unit-charge kinetic ion species, the zero-wavenumber modified-adiabatic constraint has only a global gauge degeneracy. This differs from the local kinetic-species constraint below.

Proposition 9.1 (Removal of a one-dimensional gauge)**.** *Use periodic variable* B, β, $J = 1$, $\Omega = 0$, *and* $C = 2\mathrm{Id} - P_{\mathrm{av}}$. *Then*

$$\ker M = \mathrm{span}\{F\}, \qquad M|_{\{F\}^\perp} \geq \tfrac{1}{2}\mathrm{Id}.$$

Both M *and* T_0 *reduce* $\{F\}^\perp$. *On that space* $M \geq \frac{1}{2}\mathrm{Id}$, *so the evolution and stationary-projection results apply after the constant gauge is fixed to zero.*

Proof. Since $R^*R = \mathrm{Id}$, $P_R = RR^*$ and $P_F = RP_{\mathrm{av}}R^*$ are orthogonal projections onto the local Maxwellian range and the constant-profile Maxwellian line. Their ranges are nested, and $M = \mathrm{Id} - \frac{1}{2}P_R - \frac{1}{2}P_F$. This gives the kernel and lower bound. Moreover $T_0F = 0$ because $XF = 0$; skew-adjointness makes $\{F\}^\perp$ reducing. Replace R by its projection onto this complement when using the abstract field formulas. The induced field has zero spatial mean, and its energy operator is exactly the displayed restriction. □

9.2 The kinetic-species charge and current constraints

Let B and β be constant, $J_s = 1$, and $\Omega_s = 0$ on a spatial circle of length 2π. Use kinetic species with $C_0 = \sum_s q_s^2 > 0$ and the Maxwellians of (4.1). The velocity Hilbert space is $\mathcal{H}_v = \bigoplus_s L^2(\mathrm{d}^3v_s/F_s)$. Define

$$r_s = \frac{q_sF_s}{\sqrt{C_0}}, \quad Vh = (\beta v h_s)_s, \quad w = Vr, \quad a = \|w\|_v^2 > 0, \quad E_r h = r\langle r, h\rangle_v. \tag{9.1}$$

Then $\|r\|_v = 1$, $\langle r, w\rangle_v = 0$, and $r \in \mathcal{D}(V^j)$ for every integer $j \geq 0$. The energy operator is $M_0 = \mathrm{Id} - E_r$. For a spatial Fourier mode $n \neq 0$, write $h = u + rc$ with $u = M_0h \perp r$. The equation is

$$(\mathrm{Id} - E_r)\partial_t h = -\mathrm{i}nVh. \tag{9.2}$$

Taking its r component imposes $\langle w, u\rangle_v = 0$. Thus both charge density and charge current vanish on every nonconstant spatial mode.

Set

$$K = \{r, w\}^\perp, \quad P_K : \mathcal{H}_v \to K, \quad H_K = P_KV|_{\mathcal{D}(V)\cap K}, \quad \ell(u) = a^{-1}\langle Vw, u\rangle_v. \tag{9.3}$$

The functional ℓ is bounded on $\mathcal{H}_v$ because $Vw = V^2r \in \mathcal{H}_v$.

Theorem 9.2 (The continuous-velocity constrained evolution)**.** *The operator H_K is self-adjoint on K. For every nonzero Fourier mode, the unique evolution of $u \in K$ selected by* (9.2) *is*

$$u_n(t) = e^{-\mathrm{i}ntH_K}u_n(0), \qquad h_n(t) = u_n(t) - r\ell(u_n(t)). \tag{9.4}$$

For $u_n(0) \in \mathcal{D}(H_K)$ this is the unique classical constrained solution of (9.2)*; for arbitrary $u_n(0) \in K$ it defines the energy-continuous extension. Its conserved energy is $\|u_n(t)\|_v^2$.*

Globally, the physical energy space is the closed subspace

$$\mathcal{K} = \left\{u \in L^2(\mathbb{T}; \mathcal{H}_v) : u_0 \perp r, \quad u_n \in K \ (n \neq 0)\right\}. \tag{9.5}$$

The generator has domain

$$\left\{u \in \mathcal{K} : u_n \in \mathcal{D}(H_K) \ (n \neq 0), \quad \sum_{n\neq 0} n^2\|H_Ku_n\|_v^2 < \infty\right\}, \tag{9.6}$$

acts as $-\mathrm{i}nH_K$ on $n \neq 0$, and is zero on $n = 0$. It generates a unitary group on $\mathcal{K}$. Finite spatial Fourier sums with projected Maxwellian polynomial velocity data form a dense core. The remaining spatially constant charge component $rc_0(t)$ is a gauge and is set to zero.

Proof. The finite-rank projection $\mathrm{Id} - P_K$ has range spanned by $r, w \in \mathcal{D}(V)$. It preserves $\mathcal{D}(V)$, and the off-diagonal blocks $P_KV(\mathrm{Id} - P_K)$ and $(\mathrm{Id} - P_K)VP_K$ extend to bounded finite-rank operators. Subtracting these adjoint blocks from the self-adjoint multiplication operator V gives a self-adjoint

block-diagonal operator with domain $\mathcal{D}(V)$. Its restriction to the reducing subspace K is H_K, proving self-adjointness.

For a classical solution, differentiating the current constraint gives

$$0 = \langle w, \partial_t u\rangle_v = -\mathrm{i}n\{\langle w, Vu\rangle_v + ac\}.$$

Thus $c = -\ell(u)$. Since $u \in K$,

$$V(u - r\ell(u)) = Vu - w\ell(u) = P_K V u = H_K u.$$

Inserting this identity into (9.2) proves (9.4), including uniqueness for classical data. Self-adjointness extends the solution unitarily to every $u \in K$.

The orthogonal direct sum over Fourier modes is skew-adjoint with the domain (9.6). Multiplication by v has the Maxwellian polynomial functions as a core. Projecting them by P_K preserves graph convergence: the removed components lie in the fixed finite-dimensional space spanned by r, w. Finite Fourier truncation then proves the global core assertion. For $n = 0$, (9.2) only says that u_0 is constant in time and imposes no current constraint. The component $rc_0(t)$ has zero mass and zero transport; quotienting it, or fixing it to zero, removes this gauge freedom. □

The additional current constraint is essential. Differentiating it once determines the charge component c; treating M_0^{-1} as an ordinary inverse would miss both steps.

9.3 Approximation by coercive energy operators

Consider the specified regularization

$$M_\varepsilon = \mathrm{Id} - (1-\varepsilon)E_r, \qquad 0 < \varepsilon \le 1. \tag{9.7}$$

It is obtained by replacing C_0 by $C_0/(1-\varepsilon)$ when $\varepsilon < 1$ and leaving $J_s = 1$. Thus it regularizes the homogeneous field constraint. It is not an identification with every small-radial-wavenumber Bessel model. In the energy coordinate $y = M_\varepsilon^{1/2} h$, the mode Hamiltonian is

$$H_\varepsilon = M_\varepsilon^{-1/2} V M_\varepsilon^{-1/2} = V + (\varepsilon^{-1/2} - 1)(E_r V + V E_r). \tag{9.8}$$

The second equality uses $E_r V E_r = 0$. The perturbation is bounded and finite rank for each fixed ε, so H_ε is self-adjoint on $\mathcal{D}(V)$.

Theorem 9.3 (Convergence in the energy coordinate)**.** *There is a constant C, independent of n, t, ε, such that for every $u \in K$, every nonzero integer n, and $t \in \mathbb{R}$,*

$$\|e^{-\mathrm{i}ntH_\varepsilon} u - e^{-\mathrm{i}ntH_K} u\|_v \le C\sqrt{\varepsilon}(1 + |nt|)\|u\|_v. \tag{9.9}$$

Consequently the corresponding spatial groups converge strongly, uniformly on compact time intervals, for every $u \in \mathcal{K}$. If $u \in H^1(\mathbb{T}; \mathcal{H}_v) \cap \mathcal{K}$, then for $|t| \le T$ their difference is at most

$$C\sqrt{\varepsilon}(\|u\| + T\|\partial_\theta u\|). \tag{9.10}$$

These limits concern $y = M_\varepsilon^{1/2} h$. They do not prove convergence of the unreduced state h, its charge component, or its field.

Proof. For $u \in K$ define the bounded corrector $S_\varepsilon u = u - \sqrt{\varepsilon} r \ell(u)$. On $\mathcal{D}(H_K)$, (9.8) gives

$$H_\varepsilon u = Vu, \quad H_\varepsilon r = w/\sqrt{\varepsilon}, \qquad H_\varepsilon S_\varepsilon u = H_K u. \tag{9.11}$$

In particular,

$$H_\varepsilon S_\varepsilon - S_\varepsilon H_K = \sqrt{\varepsilon} r \ell H_K.$$

The last operator is bounded from K to $\mathcal{H}_v$. Indeed,

$$\ell(H_K u) = a^{-1} \langle H_K P_K V w, u \rangle_v,$$

where $P_K V w \in \mathcal{D}(H_K)$ because $r \in \mathcal{D}(V^3)$. Duhamel's formula, applied first on $\mathcal{D}(H_K)$ and then by density, yields

$$\| e^{-\mathrm{i}ntH_\varepsilon} S_\varepsilon u - S_\varepsilon e^{-\mathrm{i}ntH_K} u \|_v \le \sqrt{\varepsilon} |nt| \, \| \ell H_K \| \, \| u \|_v.$$

The two errors from replacing S_ε by the identity are each at most $\sqrt{\varepsilon} \|\ell\| \|u\|_v$. This proves (9.9). Summing the squared Fourier estimates and using the triangle inequality in ℓ^2 gives (9.10). For general $u \in \mathcal{K}$, approximate by finite Fourier sums and use the unitary bound for both groups. The zero mode is unchanged throughout. □

Proposition 9.4 (Why unrestricted data cannot have the same limit)**.** *The full energy-coordinate groups do not converge to a strongly continuous group uniformly on compact time intervals for all initial data. For initial vector r in mode $n = 1$ and $t_\varepsilon = s\sqrt{\varepsilon}$,*

$$e^{-\mathrm{i}t_\varepsilon H_\varepsilon} r \longrightarrow \cos(s\sqrt{a}) r - \mathrm{i} \sin(s\sqrt{a}) \, w/\sqrt{a}. \tag{9.12}$$

Proof. Put $K_f = E_r V + V E_r$. It is bounded, maps r to w and w to ar, and leaves their span invariant. Also $\sqrt{\varepsilon} H_\varepsilon = K_f + \sqrt{\varepsilon}(V - K_f)$. For r the orbit $e^{-\mathrm{i}sK_f} r$ lies in the fixed span of r, w, on which $V - K_f$ is bounded. Duhamel's formula therefore bounds the difference from $e^{-\mathrm{i}sK_f} r$ by a constant times $\sqrt{\varepsilon}|s|$ on compact s intervals. Exponentiating the two-dimensional matrix gives (9.12). For a generic fixed s, its limit differs from r although $t_\varepsilon \to 0$. Uniform convergence to a strongly continuous group would force that difference to vanish. □

Theorems 9.2 and 9.3 resolve the domain and approximation questions in the homogeneous model. Variable magnetic geometry, a nonzero orbit drift, and a simultaneous Bessel-gyroaverage limit change the constraint and require separate estimates.

10 A stationary-response calculation with trapped and passing particles

Consider one unit-mass, unit-charge species on $[-\pi, \pi]$, with

$$B = 2 - \cos\theta, \quad \beta = 1, \quad \Omega = \alpha\mu \sin\theta, \quad J = J_0\!\left(k_x \sqrt{2\mu/B}\right), \quad C = 2\mathrm{Id} - P_{\mathrm{av}}, \tag{10.1}$$

where $\alpha = 0.6$, $k_x = 0.8$, and P_{av} is the orthogonal average for $\mathrm{d}\theta/B$. Set $h_0 = Rf$ with $f = 1 + 0.4\cos(2\theta) + 0.2\sin\theta$. Since $Xv = -\mu \sin\theta$, the phase $\Psi = -\alpha v$ has $X\Psi = \Omega$. The orbit mean drift is therefore zero. Moreover

$$\Gamma(\theta) = R^* R = e^{-b} I_0(b), \quad b = k_x^2/B^2, \qquad M \ge m_0 \mathrm{Id}, \quad m_0 = 1 - e^{-k_x^2/9} I_0(k_x^2/9) > 0.$$

Here $m_0 = 0.0674638$ to the displayed precision. The bound follows from $C^{-1} \le \mathrm{Id}$, $\|R\|^2 = \max \Gamma$, and the monotonicity of $e^{-b} I_0(b)$ for $b > 0$, obtained by differentiating its integral representation.

Let $A_o = R^* \Pi R$. The correct and the ordinary-orbit-average fields solve, respectively,

$$(C - A_o)\phi_\infty = A_o(f - C^{-1}\Gamma f), \qquad C\phi_{\mathrm{orb}} = A_o f. \tag{10.2}$$

The second field is $Q\Pi h_0$, not $QP_M h_0$. No other modeling difference enters the comparison.

10.1 Orbit quadrature and field approximation

The passing pitches are $0 < \lambda < 1/3$, with both orientations included; trapped pitches are $1/3 < \lambda < 1$. Write $a_\lambda = 1 - \lambda$ and $K = K(k)$, with K the complete elliptic integral. Passing orbits use

$$k^2 = 2\lambda/a_\lambda, \quad 0 < u < 2K, \quad \theta = 2\,\mathrm{am}(u,k), \quad v = \pm\sqrt{2Ea_\lambda}\,\mathrm{dn}(u,k), \quad \mathcal{T} = 2\sqrt{2}K/\sqrt{a_\lambda}.$$

Trapped orbits use

$$k^2 = a_\lambda/(2\lambda), \quad 0 < u < 4K, \quad \theta = 2\arcsin(k\,\mathrm{sn}(u,k)), \quad v = \sqrt{2Ea_\lambda}\,\mathrm{cn}(u,k), \quad \mathcal{T} = 4K/\sqrt{\lambda}.$$

Differentiating these formulas verifies the characteristic equations; u is proportional to travel time. The orbit averages use equally spaced midpoint nodes in u. Pitch quadrature applies Gauss–Legendre nodes after $\lambda = (1 - x^3)/3$ for passing particles and $\lambda = 1/3 + 2x^3/3$ for trapped particles. Energy quadrature uses generalized Gauss–Laguerre nodes with weight $\sqrt{E}e^{-E}$. The normalized field-form weight is $\mathcal{T}\sqrt{E}e^{-E}/\sqrt{2\pi}$ for each orbit orientation.

Both field equations are assembled in the real trigonometric basis through mode 12, of dimension 25. The spatial inner products use 8192 equally spaced midpoint nodes. Table 1 varies all orbit quadrature orders together. The last two grids change the correct field by 1.65×10^{-9} in relative $\mathcal{X}$ norm. The ordinary-orbit-average field differs from it by 0.231558, or 23.2%. The volume check compares the assembled Maxwellian orbit volume with the exact $\int_{-\pi}^{\pi} B^{-1}\,\mathrm{d}\theta = 2\pi/\sqrt{3}$.

Table 1: Refinement of the specified orbit calculation. The three orders are pitch, energy, and travel-time nodes. Errors compare computed fields in the same 25-dimensional space.

Orders	Mean of ϕ_∞	Relative field change	Orbit-average error	Volume error
$24, 24, 128$	0.5752662305	—	0.2315581501	$4.68\,10^{-9}$
$40, 40, 256$	0.5752662461	$2.31\,10^{-7}$	0.2315581265	$2.28\,10^{-10}$
$64, 56, 384$	0.5752662468	$1.65\,10^{-9}$	0.2315581254	$1.40\,10^{-11}$

Increasing the field basis from mode 10 to mode 12 changes the field by 3.49×10^{-5} relative to the mode-12 result; the full sequence is included in the generated tables. These differences diagnose discretization convergence but do not bound the continuum error. The calculation neither evolves the collisionless system nor identifies the reported percentage with a Rosenbluth–Hinton residual amplitude.

10.2 Checks of the singular-limit and resolvent identities

The continuous-velocity argument is supplemented by Hermite Jacobi matrices for velocity multiplication, of sizes 16, 32, and 64. With the first two Hermite vectors spanning r, w, all 12 tested size

and regularization pairs satisfy (9.11) with maximum residual 2.23×10^{-16}. At time 0.7 and mode one, the largest ratio of the measured group error to the explicit corrector bound is 0.508. For size 64 the group errors at $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}$ are, respectively,

$$0.133375, \qquad 0.0133207, \qquad 0.00230482, \qquad 0.000143805.$$

Oscillatory prefactors need not make the error a monotone multiple of $\sqrt{\varepsilon}$. These are finite-matrix consistency checks; the continuum domain and convergence statements are proved in Section 9.

Fifty complex matrices test the Woodbury identity (3.2) and positive-axis coercivity; the largest relative identity error is 1.62×10^{-14}. Additional cases test the stationary projection and the failure of operator-norm averaging. The scripts record all parameters and random seeds. These matrix calculations check the implemented identities, not the continuum proofs or a kinetic solver.

11 Consequences and remaining questions

When the energy operator is strictly positive, the stationary field equation remains stable even without uniform kinetic relaxation. Its residual estimate requires control of assembly error, not merely an accurate matrix solve. The orbit calculation shows that energy projection and ordinary orbit averaging can produce substantially different fields.

The two singular limits have different origins. Drift detuning changes the kernel while the energy operator remains invertible; averaging describes the former stationary states on the inverse-detuning scale. Vanishing polarization instead changes the admissible state space. The homogeneous constraint theorem treats that endpoint for the specified regularization. Variable geometry, simultaneous limits in the gyroaverage, and unbounded collision operators require additional domain estimates.

The ordinary long-time field limit is also distinct from the averaged response. Proving it would require information about the coupled spectral measure or a limiting-absorption estimate, neither of which follows from the orbit spectral density or scalar continuation developed here. The embedded-eigenvalue criterion is necessary and sufficient, but injectivity of its field equation is not established in general. Thus orbit resonances, zeros of a continued scalar form, and eigenfunctions of the coupled generator remain separate questions.

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