SHARP REACH BOUNDS AND QUANTITATIVE LOCALIZATION FOR OPAQUE SETS

MANTIS DISCOVERY

ABSTRACT. Let Q be the unit square and put $\Lambda = \sqrt{2} + \sqrt{6}/2$. We prove two sharp inequalities for finite-length continua in the plane. A set containing two adjacent corners has length, plus its convex-hull distances to the other corners, at least Λ. A corresponding inequality holds for two sets whose hulls meet and which together contain three corners. These bounds extend to finite chains of component hulls, with coefficient one on the total distance between consecutive hulls. The argument also allows missing projection intervals and applies to every closed opaque set of finite length, regardless of its number of components or its location. After the corners are adjoined, any such barrier shorter than Λ has either four distinct corner components or one component containing an adjacent pair and two further components, one through each remaining corner. In the latter case, we obtain positive lower bounds on the path distances between the three hulls. A tree replacement argument rules out some configurations in the former case. Exact arithmetic verifies the algebraic inequalities for all 105 six-leaf topologies and all 45 five/four-leaf pairs. The other configurations, and the unrestricted lower bound for the square, remain open.

1. INTRODUCTION AND RESULTS

A set $F \subset \mathbb{R}^2$ is *opaque* for a convex body Q if every line meeting Q also meets F. Its length is $\ell(F) = \mathcal{H}^1(F)$. The barrier may lie outside Q and need not be connected. For the unit square, the shortest known construction has length

$$\Lambda = \sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389584337\ldots. \tag{1}$$

This construction is classical; see [2, Fig. 1]. For rectifiable barriers, Kiderlen and Pausinger [6] proved the lower bound $2 + 6.3 \cdot 10^{-5}$, improving the bound of Kawamura, Moriyama, Otachi and Pach [5]. The gap between that bound and (1) remains large.

We study the geometric restrictions imposed by a length below Λ. The argument retains the convex hull of each connected component, rather than replacing a collection of components by its aggregate hull. Two sharp inequalities account for the distance that selected components must still reach. A projection interval missed by those components must be covered by the remainder. Only finitely many components enter each application, so the remainder can have infinitely many components and can be unbounded.

Write

$$Q = [0,1]^2, \quad A = (0,0), \quad B = (1,0), \quad C = (0,1), \quad D = (1,1).$$

Throughout, a *continuum* is a nonempty compact connected subset of $\mathbb{R}^2$. All continua used below have finite length. Such sets are rectifiable; see [1, Theorem 4.4]. A singleton is allowed.

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Theorem 1.1 (Sharp reach inequalities)**.** *Let E, E_0, E_C be continua of finite length. If $A, B \in E$, then*

$$\ell(E) + \operatorname{dist}(C, \operatorname{conv} E) + \operatorname{dist}(D, \operatorname{conv} E) \geq \Lambda. \tag{2}$$

If $A, B \in E_0$, $C \in E_C$, and $\operatorname{conv} E_0 \cap \operatorname{conv} E_C \neq \varnothing$, then

$$\ell(E_0) + \ell(E_C) + \operatorname{dist}\big(D, \operatorname{conv}(E_0 \cup E_C)\big) \geq \Lambda. \tag{3}$$

Both constants are attained. No containment in Q is required.

The proofs reduce these inequalities to balanced vector loads on Steiner trees. The required load systems depend on the tree topology. We cover all 105 labeled six-leaf topologies for (2); for (3), three explicit families cover 39 of the 45 five/four-leaf topology pairs, and an exact rectangle cover handles the remaining six. The geometric reductions are proved in section 4. The load families, analytic estimates, and verification procedure are in section A.

1.1. **Distances between component hulls.** Let $F \subset \mathbb{R}^2$ be closed with $\ell(F) < \infty$, and adjoin the corners:

$$\widehat{F} = F \cup \{A, B, C, D\}.$$

This does not change length and preserves opacity. Every connected component E_i of $\widehat{F}$ is compact: it is closed, and its diameter is at most its finite length. Set $K_i = \operatorname{conv} E_i$ and

$$g(i, j) = \operatorname{dist}(K_i, K_j).$$

The shortest-path pseudometric associated with these distances is

$$d_F(i, j) = \inf \left\{ \sum_{r=1}^{k} g(i_{r-1}, i_r) : i_0 = i,\ i_k = j,\ i_0, \ldots, i_k \text{ is a finite chain} \right\}. \tag{4}$$

We write $d_F^{\setminus i_0}$ when chains may not use the component E_{i_0}. We set $d_F(i, i) = 0$. In chains between distinct vertices, repetitions can be removed. The ordinary hull-intersection graph has an edge precisely when $g(i, j) = 0$; its paths therefore have zero cost in (4).

For a unit vector u, let $p_u(x) = u \cdot x$. For compact K, write $h_K(u) = \max_{x \in K} u \cdot x$. Define the maximum missing projection length by

$$\eta(F) = \sup_{u \in S^1} \big| p_u(Q) \setminus p_u(F) \big|, \tag{5}$$

where $|\cdot|$ denotes one-dimensional Lebesgue measure. These projections are measurable: a closed subset of $\mathbb{R}^2$ is a countable union of compact sets. Adding the four corners does not change $\eta(F)$. An opaque set has $\eta(F) = 0$.

Theorem 1.2 (Quantitative localization)**.** *Let $F \subset \mathbb{R}^2$ be closed and of finite length.*

(1) *If a component of $\widehat{F}$ contains three corners, or two opposite corners, then*

$$\ell(F) + \eta(F) \geq \Lambda. \tag{6}$$

(2) *If distinct components contain A, B and C, D, respectively, then*

$$\ell(F) + \eta(F) \geq 1 + \sqrt{3}. \tag{7}$$

(3) *Suppose the corner partition is $AB \mid C \mid D$, with components E_0, E_C, E_D. Then*

$$\ell(F) + \eta(F) + d_F(E_0, E_C) \geq \Lambda, \tag{8}$$

$$\ell(F) + \eta(F) + d_F(E_0, E_D) \geq \Lambda, \tag{9}$$

$$\ell(F) + \eta(F) + d_F^{\setminus E_0}(E_C, E_D) \geq 1 + \sqrt{3}. \tag{10}$$

The conclusions are invariant under the symmetries of the square.

Here and below, $d_F(E_i, E_j)$ means the distance between the corresponding component indices. In particular, if $\delta = \Lambda - \ell(F) - \eta(F) > 0$ in part (3), then

$$\begin{aligned} d_F(E_0, E_C),\ d_F(E_0, E_D) &\geq \delta, \\ d_F^{\setminus E_0}(E_C, E_D) &\geq 1 + \sqrt{3} - \Lambda + \delta. \end{aligned} \tag{11}$$

Thus the conclusion records a metric separation, not just the absence of a finite path of intersecting hulls.

Corollary 1.3 (Closed opaque sets)**.** *If F is a closed opaque set for Q with $\ell(F) < \Lambda$, its augmented corner partition, up to a symmetry of Q, is one of the following:*

$$A \mid B \mid C \mid D, \qquad AB \mid C \mid D.$$

In the second case the three corner components lie in distinct connected components of the full hull-intersection graph and satisfy (11) *with $\delta = \Lambda - \ell(F)$.*

The proof does not approximate F by a finite barrier. It uses finitely many of its actual components and charges their complement through one projection. This distinction allows theorem 1.3 to include closed barriers with infinitely many components, including a nonrectifiable remainder. Closedness, rather than finiteness or boundedness, supplies the compact components needed by the argument.

1.2. **Relation to earlier work.** Replacing a graphical component by a Steiner tree spanning its hull vertices is already present in Provan, Brazil, Thomas and Weng [7, Lemma 2.10]. They also prove the connected triangle result [7, Theorem 4.11]. We use a short dual proof of the associated three-point inequality. For general polygons, the theorem identifying a minimum connected barrier with a Steiner tree of the polygon vertices in Dumitrescu, Jiang and Pach [2, Theorem 5] requires the barrier to be interior. Neither (2) nor (3) uses that restriction.

The projection estimates of Jones [4], the multiplicity method of [5, 6], and the orientation stability theorem of Steinerberger [8] concern length near the half-perimeter bound. Our estimates instead use prescribed component incidence and retain metric distances between convex hulls. They do not improve the unrestricted numerical lower bound. The four-singleton pattern and the separated adjacent pattern in theorem 1.3 remain. In section 6 we give an additional finite-tree criterion for the former; in section 7 we compute (5) exactly for rational segment unions.

2. Connected sets and finite trees

For a finite list $P = (P_1, \ldots, P_m)$, let $\mathrm{st}(P)$ denote the length of a Euclidean Steiner minimal tree spanning its entries. Repeated entries are permitted. Standard Steiner-tree theory gives existence, straight edges, and degree three with 120° angles at unprescribed branch points [3]. In particular,

$$\mathrm{st}(P) \leq \ell(E) \qquad (P_1, \ldots, P_m \in E) \tag{12}$$

for every finite-length continuum E. This comparison is with the length of the set, not with the length of a parametrization that may traverse an edge twice.

Lemma 2.1 (Projection and remainder estimates)**.** *For a continuum E and $u \in S^1$,*

$$p_u(E) = p_u(\operatorname{conv} E) = [\min_E p_u, \max_E p_u]. \tag{13}$$

Let F be closed and let $X \subset F$ be a finite union of compact sets. If a measurable set $I \subset p_u(Q)$ is disjoint from $p_u(X)$, then

$$\ell(F \setminus X) \geq |I| - \eta(F). \tag{14}$$

Proof. A continuous image of a connected set in $\mathbb{R}$ is an interval, and compactness supplies its endpoints. Linear functionals have the same extrema on a set and its convex hull, proving (13). Every point of $I \cap p_u(F)$ belongs to $p_u(F \setminus X)$. Orthogonal projection is 1-Lipschitz, so

$$\ell(F \setminus X) \geq |p_u(F \setminus X)| \geq |I| - |p_u(Q) \setminus p_u(F)| \geq |I| - \eta(F).$$

The sets involved are countable unions of compact sets, so their projections are measurable. □

Lemma 2.2 (Three terminals in a convex hull)**.** *If $P_1, P_2, P_3 \in \operatorname{conv} E$ for a finite-length continuum E, then*

$$\operatorname{st}(P_1, P_2, P_3) \leq \ell(E). \tag{15}$$

A realizing tree can be chosen inside $\operatorname{conv} E$*. The same statement holds for one or two prescribed points, by repetition.*

Proof. The three-terminal Steiner problem is the Fermat problem. Its convex dual is

$$\min_{z \in \mathbb{R}^2} \sum_{i=1}^{3} |P_i - z| = \max \left\{ \sum_{i=1}^{3} v_i \cdot P_i : \sum_i v_i = 0,\ |v_i| \leq 1 \right\}. \tag{16}$$

For completeness, the inequality from right to left follows by inserting z and applying Cauchy–Schwarz. At a minimizing z, the subgradient condition for the sum of distances supplies vectors v_i of norm at most one, summing to zero, for which $v_i \cdot (P_i - z) = |P_i - z|$. This proves equality.

Choose maximizing loads v_i and points $z_i \in E$ maximizing $v_i \cdot x$. Since $P_i \in \operatorname{conv} E$,

$$\operatorname{st}(P_1, P_2, P_3) = \sum_i v_i \cdot P_i \leq \sum_i v_i \cdot z_i \leq \operatorname{st}(z_1, z_2, z_3) \leq \ell(E).$$

Nearest-point projection onto the closed convex set $\operatorname{conv} E$ is 1-Lipschitz and fixes the terminals. Projecting a minimizing tree therefore keeps it inside that hull without increasing its length. □

The three-terminal restriction matters. Let E be the Steiner tree of an equilateral triangle, and add to its three vertices the midpoint of one side. All four points lie in $\operatorname{conv} E$, but their Steiner tree has length strictly greater than $\ell(E)$. Otherwise a minimizer spanning the four points would also be the unique Steiner tree of the original triangle, which does not contain that midpoint. We will therefore prescribe at most three virtual terminals in any component that is replaced.

Lemma 2.3 (Chords and exposed faces)**.** *If E is a planar continuum, every point of* $\operatorname{conv} E$ *lies on a segment joining two points of E. If q is the nearest point to $P \notin \operatorname{conv} E$, these two points may be chosen as the endpoints of the exposed face of* $\operatorname{conv} E$ *with normal $P - q$.*

Proof. The assertion is immediate when the point z belongs to E. Otherwise radial projection about z maps E continuously to a compact connected subset of the unit circle. If that image contained no antipodal pair, it would be contained in an open semicircle. A line through z would then strictly separate z from E, contradicting $z \in \operatorname{conv} E$. An antipodal pair gives the required chord.

For the second assertion, the nearest-point inequality makes the line through q perpendicular to $P - q$ a supporting line. Its face is a segment or a singleton. The endpoints are extreme points of $\operatorname{conv} E$ and belong to E, since E is compact. They span q. □

Lemma 2.4 (Tree load inequality)**.** *Assign vectors b_i to the terminals P_i of a finite embedded tree, with $\sum_i b_i = 0$. If the sum of the loads on either side of every edge has norm at most one, then*

$$\sum_i b_i \cdot P_i \leq \sum_e |e|. \tag{17}$$

Proof. Root the tree and orient its edges away from the root. For an edge e let B_e be the sum of the terminal loads in the component containing its head after e is deleted. Summation along root-to-terminal paths gives

$$\sum_i b_i \cdot P_i = \sum_e B_e \cdot \left(x_{\mathrm{head}(e)} - x_{\mathrm{tail}(e)}\right) \le \sum_e |e|.$$

The root term vanishes because the loads balance. □

A Steiner tree on $m \ge 3$ labeled terminals can be refined to a full unrooted binary tree by adding zero-length edges. A terminal lying at an internal vertex receives a zero-length leaf; coincident labels are separated in the same way. Resolve any resulting higher-degree vertex by a binary tree of zero-length edges, then suppress unmarked degree-two vertices. The embedded length is unchanged. Thus it suffices to check all $(2m-5)!!$ full labeled binary topologies, including the cuts introduced by this refinement. We will use 3, 15, and 105 topologies at $m = 4, 5, 6$.

3. Elementary reach estimates

Put

$$n = \frac{(1,1)}{\sqrt{2}}, \qquad m = \frac{(1,-1)}{\sqrt{2}}, \qquad c = \cos\frac{\pi}{12}, \quad s = \sin\frac{\pi}{12}.$$

Then $c + s = \sqrt{6}/2$. The following estimates will handle the corner partitions that do not require the two main inequalities.

Lemma 3.1. *For finite-length continua, the following statements hold.*

(1) *If $A, B, C, z \in E$, then*

$$\ell(E) \ge n \cdot z + \frac{\sqrt{6}}{2}. \tag{18}$$

(2) *If $A, D \in E$, $a = \max_E m \cdot x$, and $b = -\min_E m \cdot x$, then*

$$\ell(E) \ge a + b + \frac{\sqrt{6}}{2}. \tag{19}$$

(3) *If $A, B \in E$ and $h = \max_E x_2$, then*

$$\ell(E) \ge h + \frac{\sqrt{3}}{2}. \tag{20}$$

Proof. For part (1), take a Steiner tree on A, B, C, z and refine it to a full four-leaf topology. The following balanced loads, in the order A, B, C, z, have unit norms and admissible internal cuts:

split	loads	work
$AB \mid Cz$	$(-n, (c,-s), -(c,-s), n)$	$n \cdot z + c + s$
$AC \mid Bz$	$(-n, -(-s,c), (-s,c), n)$	$n \cdot z + c + s$
$Az \mid BC$	$(-n, m, -m, n)$	$n \cdot z + \sqrt{2}$.

In the first two rows the designated pair sum has norm one; in the last it is zero. Apply theorem 2.4 and use $\sqrt{2} > \sqrt{6}/2$.

For part (2), choose $z_+, z_- \in E$ realizing $a, -b$. Give them loads $m, -m$, and give A, D loads $-u, u$. For the splits $AD \mid z_+z_-$, $Az_+ \mid Dz_-$, and $Az_- \mid Dz_+$, choose respectively

$$u = n, \qquad u = (c,s), \qquad u = (s,c).$$

The internal cut has norm $0, 1, 1$, respectively. The work is $a+b+\sqrt{2}$ in the first case and $a+b+c+s$ in the other two.

For part (3), take a maximum-height point $z \in E$. The loads

$$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad (0, 1)$$

at A, B, z are balanced and have unit norms. The three-terminal dual formula or theorem 2.4 gives (20). □

Proof of theorem 1.2(1)–(2). We work with $\widehat{F}$, writing it as F; both length and $\eta(F)$ are unchanged. Suppose E contains A, B, C, and take $z \in E$ maximizing $n \cdot x$. The target projection is $[0, \sqrt{2}]$. Its part beyond $p_n(E)$ has length $(\sqrt{2} - n \cdot z)_+$ and must be covered by the remainder, except for at most $\eta(F)$. By theorems 2.1 and 3.1,

$$\ell(F) + \eta(F) \geq n \cdot z + \frac{\sqrt{6}}{2} + (\sqrt{2} - n \cdot z)_+ \geq \Lambda.$$

For an opposite pair, use the two disjoint tails of $[-1/\sqrt{2}, 1/\sqrt{2}]$ outside $[-b, a] = p_m(E)$. Their combined length is $(1/\sqrt{2} - a)_+ + (1/\sqrt{2} - b)_+$. One application of (14) to their union loses at most $\eta(F)$, not twice that quantity. Adding (19) proves (6).

For adjacent pairs, let $h = \max_{E_-} x_2$ and $k = \min_{E_+} x_2$, where $A, B \in E_-$ and $C, D \in E_+$. Reflection of (20) gives

$$\ell(E_-) + \ell(E_+) \geq h + \frac{\sqrt{3}}{2} + 1 - k + \frac{\sqrt{3}}{2}.$$

The interval (h, k), when nonempty, is contained in $[0, 1]$ and missed by both components. The remainder costs at least $(k - h)_+ - \eta(F)$. Adding these disjoint length contributions proves (7). □

4. Proof of the sharp reach inequalities

For balanced corner loads a, b, c, d, write

$$J(a, b, c, d) = b_x + c_y + d_x + d_y. \tag{21}$$

The two finite load statements below are verified in section A. They contain no geometric hypotheses about the locations of the actual terminals.

Proposition 4.1 (Six-leaf loads)**.** *Label the leaves A, c_-, c_+, d_-, d_+, B. For every full binary topology on these labels and every $(\lambda, \mu) \in [0, 1]^2$, there are vectors a, b, c, d such that*

$$a + b + c + d = 0, \qquad |a|, |b|, |c|, |d| \leq 1, \qquad J(a, b, c, d) \geq \Lambda,$$

and every edge cut has load sum of norm at most one for the leaf loads

$$a, \quad (1 - \lambda)c, \quad \lambda c, \quad (1 - \mu)d, \quad \mu d, \quad b. \tag{22}$$

Proof of (2). Put $K = \operatorname{conv} E$ and let q_C, q_D be nearest points in K to C, D. By theorem 2.3, there are actual points $c_-, c_+, d_-, d_+ \in E$ such that

$$q_C = (1 - \lambda)c_- + \lambda c_+, \qquad q_D = (1 - \mu)d_- + \mu d_+,$$

with $\lambda, \mu \in [0, 1]$. This includes zero distance: if a corner is inside K, the connected-chord assertion supplies its representation.

Choose a Steiner tree on the six actual terminals and refine its topology. Apply theorems 2.4 and 4.1 to obtain

$$\ell(E) \geq a \cdot A + b \cdot B + c \cdot q_C + d \cdot q_D.$$

Since $|c|, |d| \leq 1$,

$$c \cdot q_C + |C - q_C| \geq c \cdot C, \qquad d \cdot q_D + |D - q_D| \geq d \cdot D.$$

Adding these inequalities gives (2). □

Proposition 4.2 (Five/four-leaf loads)**.** *Let $\lambda, \mu \in [0,1]$ and $0 \le u \le v \le 1$. For every five-leaf topology on $(A, B, q_0, z_{0,-}, z_{0,+})$ and four-leaf topology on $(C, q_C, z_{C,-}, z_{C,+})$, there are balanced corner loads with*

$$|c| \le 1, \quad |d| \le 1, \quad |G_v| \le 1, \qquad G_t = c + td, \qquad J(a,b,c,d) \ge \Lambda,$$

such that all edge-cut norms are at most one under the leaf loads

$$\begin{aligned} &(a, b, (1-v)d, (1-\lambda)G_v, \lambda G_v), \\ &(c, ud, -(1-\mu)G_u, -\mu G_u). \end{aligned} \tag{23}$$

Proof of (3). Write $K_0 = \operatorname{conv} E_0$, $K_C = \operatorname{conv} E_C$, and $H = \operatorname{conv}(K_0 \cup K_C)$. Choose $z \in K_0 \cap K_C$ and a nearest point q to D in H. Put $\rho = |D - q|$.

If $q \in K_0$, then $\ell(E_C) \ge \operatorname{diam} K_C \ge |C - z| \ge \operatorname{dist}(C, K_0)$, whereas $\rho = \operatorname{dist}(D, K_0)$. Inequality (2) applied to E_0 proves the result. If $q \in K_C$, replace E_0 and E_C by trees spanning A, B, z and C, z, q, respectively, using theorem 2.2. Their union is connected, contains A, B, C, q, and costs at most $\ell(E_0) + \ell(E_C)$. Its distance from D is at most ρ, so (2) again applies.

Assume $q \notin K_0 \cup K_C$. If $\rho > 0$, the supporting line of H normal to $D - q$ meets both hulls. Its two exposed intervals bracket q; their adjacent endpoints $q_0 \in E_0$ and $q_C \in E_C$ satisfy

$$q = (1-\tau)q_0 + \tau q_C, \qquad 0 < \tau < 1. \tag{24}$$

The endpoints are actual points because they are extreme points of their respective hulls. By theorem 2.3,

$$z = (1-\lambda)z_{0,-} + \lambda z_{0,+} = (1-\mu)z_{C,-} + \mu z_{C,+}, \tag{25}$$

with all four chord endpoints in their corresponding continua. Set $u = v = \tau$.

If $\rho = 0$, then $D \in H \setminus (K_0 \cup K_C)$. Follow the ray from z through D to its exit point $q_\partial \in \partial H$, and write

$$D = (1-s)z + sq_\partial, \qquad 0 < s \le 1.$$

The point q_∂ belongs to neither individual hull, since otherwise convexity with z would put D in that hull. A supporting line at q_∂ therefore supplies actual endpoints q_0, q_C with $q_\partial = (1-\tau)q_0 + \tau q_C$. Retain (25) and set

$$u = s\tau, \qquad v = 1 - s + s\tau. \tag{26}$$

Then $0 \le u \le v \le 1$, $1 - v = s(1-\tau)$, and $v - u = 1 - s$.

In either case take Steiner trees on the five and four actual terminals, refine their topologies, and use theorem 4.2. The work at the chord endpoints is $(G_v - G_u) \cdot z = (v-u)d \cdot z$; the two other support terminals contribute $d \cdot ((1-v)q_0 + uq_C)$. At positive distance this sum is $d \cdot q$, and adding ρ gives at least $d \cdot D$. At zero distance it equals $d \cdot D$ by (26). The remaining work is $a \cdot A + b \cdot B + c \cdot C$. The total is at least $J(a,b,c,d) \ge \Lambda$. □

4.1. **Equality examples.** Let

$$S = \left(\frac{3 - \sqrt{3}}{6}, \frac{3 - \sqrt{3}}{6} \right), \qquad M = (1/2, 1/2), \qquad T = [A, S] \cup [B, S] \cup [C, S].$$

A direct calculation gives

$$\ell(T) = \frac{\sqrt{2} + \sqrt{6}}{2}, \qquad \operatorname{dist}(D, \operatorname{conv} T) = \frac{1}{\sqrt{2}}. \tag{27}$$

Taking $E = T$ proves equality in (2). Taking $E_0 = [A, S] \cup [B, S]$ and $E_C = [C, S]$ proves equality in (3).

The set $F_* = T \cup [M, D]$ is the classical barrier of length Λ. Indeed, $p_u(T) = p_u(\operatorname{conv}\{A, B, C\})$ contains $p_u(M)$ for every u. Its union with the interval $p_u([M, D])$ is therefore an interval containing

all four corner projections, and hence equals $p_u(Q)$. This proves opacity, including supporting lines. Equation (27) and $|MD| = 1/\sqrt{2}$ give its length.

5. Chains, projection errors, and closed barriers

We first give the geometric statement behind theorem 1.2(3).

Theorem 5.1 (Propagation along a chain)**.** *Let $E_0, \dots, E_k$ be finite-length continua with $A, B \in E_0$ and $C \in E_k$. Put $K_i = \operatorname{conv} E_i$, $H = \operatorname{conv}(\bigcup_{i=0}^{k} K_i)$, and $g = \sum_{i=1}^{k} \operatorname{dist}(K_{i-1}, K_i)$. Then*

$$\sum_{i=0}^{k} \ell(E_i) + g + \operatorname{dist}(D, H) \geq \Lambda. \tag{28}$$

The coefficient of g cannot be decreased in this inequality.

Proof. The case $k = 0$ follows from (2), since $C \in E_0$. Assume $k \geq 1$. For each i, choose $x_i \in K_{i-1}$ and $y_i \in K_i$ realizing the distance between those hulls. Join them by a segment of length $\operatorname{dist}(K_{i-1}, K_i)$. All these segments lie in H.

Let q be the metric foot of D on H. If $D \notin H$, the exposed face through q is the convex hull, on one line, of finitely many exposed intervals of the K_i. If q belongs to an individual hull, mark it in that hull. Otherwise choose the two adjacent endpoints that bracket q, and mark one in each of their distinct hulls. Each hull receives at most one mark.

If $D \in H$, mark D when it belongs to an individual hull. Otherwise follow the ray from $x_1 \in K_0$ through D to $q_\partial \in \partial H$ and mark one point, or two bracketing support points, as above. Preserving these marks and x_1 preserves D in the aggregate hull.

For $1 \leq i < k$, replace E_i by a tree inside K_i spanning y_i, x_{i+1}, and its possible mark. For $i = k$, use y_k, C, and its possible mark. Each list has at most three points, so theorem 2.2 bounds the cost by $\ell(E_i)$. The union T' of these trees and the connecting segments is connected, contains C, and meets K_0 at x_1. Its length is at most $\sum_{i=1}^{k} \ell(E_i) + g$.

The hull $H' = \operatorname{conv}(K_0 \cup T')$ is contained in H and retains q when $D \notin H$, or retains D when $D \in H$. Consequently $\operatorname{dist}(D, H') = \operatorname{dist}(D, H)$. Apply (3) to E_0, T'. □

To verify the sharp coefficient, let $u_C = (C - S)/|C - S|$ and choose $0 < t < |C - S|$. Take

$$E_0 = [A, S] \cup [B, S], \qquad E_1 = [S + t u_C, C].$$

Their aggregate hull is $\operatorname{conv}\{A, B, C\}$, and their lengths sum to $\ell(T) - t$. The vector u_C makes angle 120° with both $A - S$ and $B - S$, so it is a supporting normal of K_0 at S. It follows that $\operatorname{dist}(K_0, K_1) = t$. Equality therefore holds in (28) for every such t.

Proof of theorem 1.2(3). Again replace F by $\widehat{F}$, which has the same length and deficit. Choose a finite simple chain from E_0 to E_C, with aggregate hull H and cost g. If $d = \operatorname{dist}(D, H) > 0$, put $u = (D - q)/d$, where q is the metric foot. Since $A, B, C \in H$,

$$d = u \cdot D - h_H(u) \leq u \cdot (D - B) = u_2, \qquad d \leq u \cdot (D - C) = u_1.$$

Thus both coordinates of u are positive, and the interval $(h_H(u), u \cdot D]$ has length d, is contained in $p_u(Q)$, and is missed by every original component of the chain. The remainder has length at least $d - \eta(F)$ by theorem 2.1. If $d = 0$, the same lower bound for the remainder is automatic. Adding it to (28), and using the disjointness of the original components, gives $\ell(F) + \eta(F) + g \geq \Lambda$. Taking the infimum over finite chains proves (8). Reflection gives (9).

For (10), choose a finite simple chain from E_C to E_D avoiding E_0. Realize the hull distances as above, and let $k_0 = \min x_2$ over the chain hulls. Mark a minimum-height point in one hull. Replace each chain component by a tree spanning its incident connector endpoints, its corner when it is an endpoint component, and its possible mark. There are at most three terminals per component. The

connected union contains C, D, has minimum height k_0, and costs at most the total original chain length plus g.

The reflected vertical estimate gives a lower bound $1 - k_0 + \sqrt{3}/2$ for this union. The component E_0 contributes at least $h + \sqrt{3}/2$, where $h = \max_{E_0} x_2$. The remainder outside E_0 and the chain contributes at least $(k_0 - h)_+ - \eta(F)$. These are disjoint original length budgets. Their sum gives $\ell(F) + \eta(F) + g \geq 1 + \sqrt{3}$. The infimum over chains avoiding E_0 proves (10). □

Proof of theorem 1.3. Apply theorem 1.2(1)–(2) to the partition of the four adjoined corners into components. Every partition except $1+1+1+1$ and adjacent $2+1+1$ is excluded. In the latter case, a hull path from E_0 to either other root has zero cost and contradicts (8) or (9). A hull path from E_C to E_D either avoids E_0, in which case (10) applies, or passes through E_0, in which case its initial part is already excluded. The metric bounds follow by rearranging (8)–(10). □

5.1. **Limit contacts and approximate anchoring.** The pseudometric adds information when infinitely many components occur. For example, set $a_j = 2^{-j}$ and let E_j consist of the top and the two vertical sides of

$$[-a_j, a_j] \times [a_j, 3a_j], \qquad j \geq 1.$$

The set $F = \{(0,0)\} \cup \bigcup_{j\geq 1} E_j$ is compact, and its components are exactly the E_j and the origin. It has finite length $\sum_j 6a_j$. Consecutive rectangular hulls meet, but the origin lies in none of them. It is an isolated vertex in the ordinary hull-intersection graph, while

$$d_F(E_1, \{(0,0)\}) \leq \operatorname{dist}(\operatorname{conv} E_j, (0,0)) = a_j \longrightarrow 0.$$

Thus zero pseudometric distance need not be realized by a finite path of intersecting hulls. This example concerns the component geometry, not opacity for the square.

Corollary 5.2 (Approximate corner incidence)**.** *Let F be closed with finite length, and let E be one of its components. For any three distinct corners P_1, P_2, P_3,*

$$\ell(F) + \eta(F) + \sum_{r=1}^{3} \operatorname{dist}(P_r, E) \geq \Lambda. \tag{29}$$

For any two opposite corners P_1, P_2, the same assertion holds with the sum over $r = 1, 2$.

Proof. Join each P_r to a nearest point of the compact component E. The enlarged closed set F' has length at most $\ell(F) + \sum_r \operatorname{dist}(P_r, E)$ and $\eta(F') \leq \eta(F)$. A component of F' contains the prescribed corners, so theorem 1.2(1) applies. □

The coefficient of $\eta(F)$ in (6) is also sharp. Take $F = T$, the three-corner tree in (27). Its missing projection length is

$$\eta(T) = \max_{u \in S^1} (u \cdot D - h_{\operatorname{conv} T}(u))_+ = \operatorname{dist}(D, \operatorname{conv} T) = 1/\sqrt{2}.$$

Thus $\ell(T) + \eta(T) = \Lambda$.

6. A tree criterion for four corner components

Paths are not the only configurations for which the three-terminal replacement can be used. The following statement makes the local requirement explicit.

Theorem 6.1 (Replacement along a finite tree)**.** *Let $\mathcal{T}$ be a finite tree. Associate to each vertex v a finite-length continuum E_v, and put $K_v = \operatorname{conv} E_v$. Assign prescribed points to the hulls. For every edge vw, choose points $x_{v,w} \in K_v$ and $x_{w,v} \in K_w$. Let P_v be the set of prescribed points assigned to v together with all its incident points $x_{v,w}$. If $|P_v| \leq 3$ for every v, there is a connected finite network spanning all prescribed points with length at most*

$$\sum_v \ell(E_v) + \sum_{vw \in E(\mathcal{T})} |x_{v,w} - x_{w,v}|. \tag{30}$$

Proof. Use theorem 2.2 to replace each E_v by a tree in K_v spanning P_v at no greater length. If P_v is empty, choose any point in K_v. Join the two incident points for each edge of $\mathcal{T}$. The resulting union is connected and has the stated length bound by subadditivity of $\mathcal{H}^1$. □

Corollary 6.2. *Let F be closed and of finite length, with four distinct corner components in $\widehat{F}$. Let $\mathcal{T}$ be a finite tree on distinct augmented components, containing all four corner components. If*

$$\deg_{\mathcal{T}}(v) + \mathbf{1}_{\{E_v \text{ contains a corner}\}} \le 3 \qquad \text{for every } v, \tag{31}$$

then

$$\ell(F) + \sum_{vw \in E(\mathcal{T})} \operatorname{dist}(K_v, K_w) \ge 1 + \sqrt{3}. \tag{32}$$

In particular, a zero-cost tree satisfying (31) *is impossible below Λ. The same conclusion holds whenever the actual sets P_v have at most three distinct points, even if the degree condition fails because some incident points coincide.*

Proof. Choose nearest points for every edge and apply theorem 6.1. The resulting network spans the four corners. Their Steiner length is $1 + \sqrt{3}$. One can check the lower bound directly on the three four-leaf topologies: for $AB \mid CD$, use the corner loads

$$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

They have unit norms, their internal cut has norm one, and their work is $1 + \sqrt{3}$. Rotate the loads for $AC \mid BD$. For $AD \mid BC$, the loads $(-n, m, -m, n)$ give the larger value $2\sqrt{2}$. A tree with branch points $(1/2, 1/(2\sqrt{3}))$ and $(1/2, 1 - 1/(2\sqrt{3}))$ attains $1 + \sqrt{3}$. Finally, the original component lengths sum to at most $\ell(F)$. □

A simple path visiting all four corner components satisfies (31). A tree with two unmarked trivalent branch vertices and the four corner components as leaves also satisfies it. What the criterion does not cover is a component required to retain four unrelated virtual terminals. For example, a corner component of degree three, or a cornerless component of degree four, can create this obstruction. The counterexample following theorem 2.2 prevents a general four-point replacement at the same cost.

7. Exact computation of the projection deficit

For finite segment unions, the error term in theorem 1.2 can be computed without sampling directions. The decision version of opacity is already known to be polynomial-time solvable [2, Theorem 7]; the statement below also returns the maximum missing projection length.

Proposition 7.1. *Let P be a nondegenerate convex polygon with rational vertices, and let F be a finite union of segments with rational endpoints. Then*

$$\eta_P(F)^2 = \left(\max_{u \in S^1} |p_u(P) \setminus p_u(F)| \right)^2$$

is rational and can be computed exactly. With N denoting the total number of target vertices and segment endpoints, a direct algorithm uses $O(N^3 \log N)$ rational arithmetic operations.

Proof. The order of all projected vertices and endpoints changes only at a normal perpendicular to a difference of two input points. There are $O(N^2)$ such unoriented directions. Between consecutive critical directions, the order is fixed. Sorting the projected segment intervals and merging their union therefore expresses the missing length on that angular interval as

$$f(\theta) = a \cos\theta + b \sin\theta, \tag{33}$$

where $a, b \in \mathbb{Q}$. The coefficients are obtained by summing differences of input endpoints at the uncovered gaps.

The missing length is continuous in the direction, so its maximum occurs at a critical direction or at an interior stationary point of (33). At a critical direction represented by a nonzero rational vector u, the square of the value is $(au_1 + bu_2)^2/|u|^2$. At an interior maximum it is $a^2 + b^2$, provided (a, b) points into that angular interval. These are rational candidates. Their order and the angular containment tests use exact cross products; no angle evaluation is required. Sorting and merging $O(N)$ intervals on each of $O(N^2)$ angular intervals gives the stated operation count. □

The implementation `checks/projection_defect.py` returns the exact squared deficit and a maximizing normal. Its input coordinates are integers or rational strings. The tests include the square boundary, two diagonals, the three-corner triangle, and the rational benchmark obtained by replacing S with $(2113/10000, 2113/10000)$. The latter remains opaque because the projection argument in section 4 uses connectivity and corner containment, not the exact Fermat coordinates. For compact finite segment unions, zero deficit is equivalent to opacity: a nonempty missing part of a target projection has positive length since the segment projection is closed.

8. The remaining geometric cases

Theorem 1.2 is a necessary-condition theorem for a short barrier, not a lower bound for all barriers. An adjacent pair may remain separated from the other two corner components in the pseudometric, and the four-singleton pattern may fail every tree criterion in theorem 6.2. Neither class is eliminated here. The method also does not identify a minimizer or establish uniqueness of the classical construction.

For closed sets the component argument is direct, including infinitely many components. For nonclosed barriers, components need not be compact; the present corner-partition statement has no automatic extension to that class. More generally, approximating a barrier by segments need not preserve its component incidence or its hull graph. A further reduction must retain that structure, rather than only approximate total length.

Appendix A. Exact certificates for the reach inequalities

This appendix gives the algebraic part of theorems 4.1 and 4.2. The supplementary JSON files specify complete rectangle partitions and the rational loads on their constant leaves. The verification program rebuilds the topologies, checks every leaf, and checks the partition itself. Numerical optimization is used only to discover rational candidates.

A.1. **Coordinates and exact comparisons.** Use the orthonormal basis

$$e = (1, 1)/\sqrt{2}, \qquad f = (-1, 1)/\sqrt{2}, \qquad p = \sqrt{3}/2.$$

For loads written in this basis, put $j = \sqrt{2}J$. Then

$$j = b_e - b_f + c_e + c_f + 2d_e, \qquad \sqrt{2}\Lambda = 2 + 2p. \tag{34}$$

For rational loads in the (e, f) basis, j is rational. The exact tests

$$j > 2, \qquad (j - 2)^2 > 3$$

prove $J > \Lambda$. Norms are compared directly with one in rational arithmetic. Symbolic families use the quadratic field $\mathbb{Q}[p]/(p^2 - 3/4)$ with the positive root p.

For a constant load system, every cut vector is affine in each parameter separately. Its squared norm is separately convex, so checking the vertices of a parameter rectangle suffices. The polynomial families below produce norm gaps of degree at most two in each variable. We certify their nonnegativity by expressing them in the tensor Bernstein basis on the rectangle and checking that every coefficient is nonnegative. This is a sufficient condition, not an inference from sampled values.

A.2. **The six-leaf certificate.** A six-leaf full binary tree has three nontrivial edge splits. There are 25 possible nontrivial splits and exactly 105 compatible triples. The discovery program enumerates compatible triples; the verifier independently generates trees by repeatedly inserting a new leaf into an edge. Both produce the same complete set.

Four polynomial load families cover most of the parameter square:

$$\begin{aligned} L_\pm &: \quad a=-e,\ d=e,\ c=pf+qe,\ b=-c, \quad q=\pm(\mu-1/2),\\ R_\pm &: \quad c=f,\ b=-f,\ d=pe+qf,\ a=-d, \quad q=\pm(1/2-\lambda). \end{aligned} \tag{35}$$

They balance, all four corner loads have norm at most one on the unit square, and $j=2+2p$. Each tagged rectangle is accepted only after all its internal-cut norm gaps have nonnegative Bernstein coefficients.

The only nonpolynomial rule is needed for the topology with cherries $01, 23, 45$, where the labels follow (22), and its symmetric images. We use the rectangle

$$39/40 \le \lambda \le 1, \qquad 9/20 \le \mu \le 11/20. \tag{36}$$

Write $\delta = 1-\lambda$, $t=\mu-1/2$, and $F_\delta(t) = 2\lambda t + \delta(t^2+p^2)$.

Lemma A.1. *On* (36)*, the six-leaf load statement holds for cherries* $01, 23, 45$*.*

Proof. If $F_\delta(t) \le 0$, take

$$d=e, \qquad c=(te+pf)/\lambda, \qquad a=-e-\delta c, \qquad b=-te-pf. \tag{37}$$

These loads balance. The three internal-cut sums are $-e$, $e/2+pf$, and $e/2-pf$, all unit vectors. Moreover,

$$|a|^2 = 1+\frac{\delta}{\lambda^2}F_\delta(t) \le 1, \quad |b|^2 = t^2+p^2 \le \frac{301}{400}, \quad |c|^2 \le \frac{301/400}{(39/40)^2} < 1.$$

The objective satisfies

$$j-(2+2p) = (p+t)(1/\lambda-1) \ge 0.$$

If $F_\delta(t) \ge 0$, set

$$q_0 = \begin{cases} \dfrac{1-\sqrt{1-p^2\delta^2}}{\delta}, & \delta>0,\\ 0, & \delta=0, \end{cases} \qquad q=\max(t,q_0),$$

and use $a=-e$, $d=e$, $c=pf+qe$, $b=-c$. Here $0 \le q_0 \le 3\delta/4$, $0 \le q \le 1/20$, and $\delta(p^2+q_0^2)=2q_0$. The 01 cut has norm at most one because $2q-\delta(p^2+q^2) \ge 0$: the expression vanishes at q_0 and is increasing for the present range of q. The 45 cut is $(1/2+t-q)e-pf$; its e coefficient lies in $[0,1/2]$. The 23 cut is

$$(1/2-t+\lambda q)e+\lambda pf.$$

When $q=t$, we have $t \ge 0$ and its squared norm is at most $1/4+p^2=1$. When $q=q_0$, the inequality $F_\delta(t) \ge 0$ and $t^2+p^2 \le \lambda$ imply $t \ge -\delta/2$. Its squared norm is therefore at most

$$(1-\delta)^2p^2 + (1/2+5\delta/4)^2 = 1-\delta/4+37\delta^2/16 \le 1,$$

since $\delta \le 1/40 < 4/37$. The corner loads have squared norms at most 1 and $p^2+(1/20)^2<1$, and $j=2+2p$. □

Swapping labels $1,2$ replaces λ by $1-\lambda$; swapping $3,4$ replaces μ by $1-\mu$. A further symmetry reverses the leaf order and replaces

$$(\lambda,\mu) \mapsto (1-\mu, 1-\lambda), \qquad (a,b,c,d) \mapsto (Rb, Ra, Rd, Rc),$$

where $R(x,y)=(-x,y)$ in ordinary coordinates. In the (e,f) basis, R exchanges the two coordinates. These operations preserve balance, all cut norms, and the objective (34). An analytic leaf is accepted only when its transformed topology and rectangle satisfy theorem A.1.

The initial subdivisions in both coordinates are

$$0,\ 1/40,\ 19/40,\ 1/2,\ 21/40,\ 39/40,\ 1.$$

The reconstructed cover has the following verified counts:

Leaf or node type	Count
Rational constant-load leaves	1264
Polynomial-family leaves	2776
Analytic leaves	32
Binary subdivision nodes	292
Unresolved leaves	0

The count identity $1264 + 2776 + 32 = 105 \cdot 36 + 292$ is a useful consistency check; the verifier also checks every individual split and every initial cell, rather than relying on this identity.

A.3. **The five/four-leaf certificate.** For the second tree in (23), the base bounds $|c|, |d|, |G_v| \le 1$ handle all three topologies. Indeed,

$$G_u = (1 - u/v)c + (u/v)G_v \quad (v > 0),$$

and $G_u = c$ when $v = 0$. Thus $|G_u| \le 1$. The two other possible pair cuts are

$$c - (1-\mu)G_u = \mu c - (1-\mu)ud, \qquad c - \mu G_u = (1-\mu)c - \mu ud,$$

which are convex combinations of c and $-ud$. It remains to check the 15 five-leaf topologies and the base bounds.

Three families suffice for 13 of them:

(38)

	a	b	c	d
1	$-e$	$-pf + e/2$	$pf - e/2$	e
2	$-pe + f/2$	$-f$	f	$pe - f/2$
3	$-e$	$-pf - (1/2 - v)e$	$pf + (1/2 - v)e$	e.

Each has $j = 2 + 2p$. All relevant cut vectors are multi-affine in (λ, v), so the verifier checks their norms at the four vertices of the unit square. Exactly two five-leaf topologies remain: cherries $03, 24$ and $04, 23$. Including the three four-leaf topologies gives the $39 + 6$ split in theorem 4.2.

For cherries $03, 24$, the nontrivial cut sums are Y and $-b - Y$, where

$$Y = \lambda c + (1 - v + \lambda v)d. \tag{39}$$

Exchanging labels $3, 4$ handles the other topology. It suffices to find balanced loads on $[0,1]^2$ satisfying

$$|a|, |b|, |c|, |d|, |G_v|, |Y|, |b + Y| \le 1, \qquad j \ge 2 + 2p. \tag{40}$$

The constant leaves use rational loads. There are two analytic rules.

Lemma A.2. *Let $B = [\lambda_0, \lambda_1] \times [v_0, v_1] \subset [0,1]^2$. Put*

$$k_* = 1 - \frac{1367}{1000} v_1, \qquad H_* = \max\{|1/2 - \lambda_0 k_*|, |1/2 - \lambda_1|\}.$$

If $v_1 \le 1/10$, $k_* \ge 0$, and

$$3/4 + H_*^2 \le (1 - v_1)^2, \tag{41}$$

then (40) holds throughout B.

Proof. Set

$$r = 1 - v, \quad \Delta = 1 - v + v\lambda, \quad K = 1 - v(p + 1/2), \quad s = K/\Delta, \quad y = pe + f/2,$$

and take

$$b = -f, \qquad G_v = sf, \qquad d = (y - \lambda s f)/r, \qquad c = sf - vd, \qquad a = -b - c - d. \tag{42}$$

We have $k_* \le s \le 1$, since $p < 867/1000$, $K \le \Delta \le 1$, and $K > 0$. With $z = pv/r$, direct simplification gives

$$c = -ze + (1 - z)f, \qquad a = -pe + \frac{pv - 1/2 + \lambda s}{r} f.$$

Here $0 \le z < 1$, so $|c| \le 1$. Also $\lambda \le \Delta$, hence $\lambda s \le K$; together with $p > 1/2$, this bounds the f coefficient of a between $-1/2$ and $1/2$. Thus $|a| \le 1$. We already have $|b| = 1$ and $|G_v| = s \le 1$.

The interval $[\lambda_0 k_*, \lambda_1]$ contains λs, so

$$|d|^2 = \frac{p^2 + (1/2 - \lambda s)^2}{r^2} \le \frac{3/4 + H_*^2}{(1 - v_1)^2} \le 1.$$

Finally $Y = \lambda G_v + rd = y$, and $b + Y = pe - f/2$; both are unit vectors. Equation (34) gives $j = 2 + 2p$. □

Lemma A.3. *On $0 \le \lambda \le 1/4$, $1/4 \le v \le 3/4$, the third family in* (38) *satisfies* (40).

Proof. All four corner norms are at most one, and $G_v = e/2 + pf$ has unit norm. From (39),

$$Y = (1 - v + \lambda/2)e + \lambda p f, \qquad b + Y = (1/2 + \lambda/2)e + (\lambda - 1)pf.$$

Thus

$$|Y|^2 - 1 = \lambda^2 + (1 - v)\lambda + v^2 - 2v, \qquad |b + Y|^2 = 1 - \lambda + \lambda^2 \le 1.$$

The first polynomial is separately convex. At the four rectangle vertices its values are $-7/16$, $-15/16$, $-3/16$, and $-13/16$, all negative. The objective is $j = 2 + 2p$. □

The residual cover starts with breaks

$$\lambda : \ 0, 1/40, 1/4, 1/2, 3/4, 39/40, 1; \qquad v : \ 0, 1/100, 1/4, 1/2, 3/4, 1.$$

It has 44 rational leaves, 14 analytic leaves, and 28 binary splits, with no unresolved leaf. The 30 initial rectangles and their verified subdivisions cover the full closed parameter square. Together with the 13 uniform five-leaf families and the automatic four-leaf bounds, this proves theorem 4.2 on the full domain $0 \le u \le v \le 1$.

A.4. **Reproduction and proof boundary.** The `proofs` directory contains the two inputs:

`g7_reconstructed.json` `g12_reconstructed.json`.

The standard-library command

```
python checks/verify_certificates.py
```

regenerates the topology sets, reconstructs every load family, verifies balance, objective and cut norms, and checks the recursive covers. For the polynomial leaves it uses midpoint interpolation to reconstruct Bernstein coefficients, independently of the discovery program's power-basis calculation. Analytic tags are checked against the hypotheses of theorems A.1 to A.3, whose proofs above supply the corresponding continuous inequalities.

The verification record contains zero unresolved leaves and the counts reported above. The two payloads and the checking code are listed in the accompanying SHA-256 manifest. The generator uses floating-point optimization to propose loads; neither its solver status nor a decimal margin enters the proof. The exact checker validates the finite algebraic claims. It does not formally verify the geometric reductions from continua to the finite terminal systems.

References

[1] G. Alberti and M. Ottolini, *On the structure of continua with finite length and Gołąb's semicontinuity theorem*, Nonlinear Analysis **153** (2017), 35–55. doi:10.1016/j.na.2016.10.012.

[2] A. Dumitrescu, M. Jiang and J. Pach, *Opaque sets*, Algorithmica **69** (2014), 315–334. doi:10.1007/s00453-012-9735-2.

[3] F. K. Hwang, D. S. Richards and P. Winter, *The Steiner Tree Problem*, Annals of Discrete Mathematics **53**, North-Holland, Amsterdam, 1992.

[4] R. E. D. Jones, *Opaque sets of degree α*, American Mathematical Monthly **71** (1964), 535–537.

[5] A. Kawamura, S. Moriyama, Y. Otachi and J. Pach, *A lower bound on opaque sets*, Computational Geometry **80** (2019), 13–22. doi:10.1016/j.comgeo.2019.01.002.

[6] M. Kiderlen and F. Pausinger, *Explicit lower bounds for opaque sets of unit square and unit disc*, Acta Mathematica Hungarica **178** (2026), 289–303. doi:10.1007/s10474-026-01606-x.

[7] J. S. Provan, M. Brazil, D. Thomas and J. F. Weng, *Minimum opaque covers for polygonal regions*, preprint (2012), arXiv:1210.8139.

[8] S. Steinerberger, *A stability version of the Jones opaque set inequality*, Discrete & Computational Geometry (2026), published online. doi:10.1007/s00454-026-00835-9.