THE INERTIA OF MATCHING COFACTOR MATRICES

ANONYMOUS

Abstract. For a graph G of even order, let $W_{uv} = \mathrm{pm}(G - u - v)$ for $u \neq v$ and $W_{uu} = 0$. We determine the inertia of W when $G = L(H)$ and H is connected, subcubic, and has a positive even number of edges. Direct each bridge toward the side with an odd number of edges and replace its other endpoint by a leaf. The positive index equals the number of nonempty components after these detachments; a factorization using two-edge cuts gives the negative index and nullity. In particular, W has one positive eigenvalue exactly when every bridge of H is pendant.

For every connected strongly regular graph of even order n with least adjacency eigenvalue -2, we prove that W has inertia $(1, n-1, 0)$, except for C_4, whose inertia is $(1, 1, 2)$. Gaussian identities and switching arguments treat the infinite families, and exact integer calculations treat the exceptions. We also construct connected line graphs with least eigenvalue -2 and unbounded positive cofactor index, and a three-edge-connected cubic graph whose polynomial for four monomers is not Lorentzian.

1. Introduction

Let $G = (V, E)$ be a finite simple graph of even order. Write $\mathrm{pm}(G)$ for its number of perfect matchings, with $\mathrm{pm}(\varnothing) = 1$. The *matching cofactor matrix* is

$$W_{uv} = \mathrm{pm}(G - u - v) \quad (u \neq v), \qquad W_{uu} = 0. \tag{1}$$

When $\mathrm{pm}(G) > 0$, the ratio $W_{uv}/\mathrm{pm}(G)$ for an edge uv is the probability that a uniform perfect matching contains that edge. Nonedge entries count matchings after vertex deletion. We study the inertia $\mathrm{In}(W) = (n_+(W), n_-(W), n_0(W))$, counting positive, negative, and zero eigenvalues with multiplicity. We use J for an all-ones matrix, with its order indicated by a subscript when needed.

A nonzero symmetric matrix with nonnegative entries and zero diagonal has both positive and negative eigenvalues. The smallest possible positive index is therefore one. For W, this is equivalent to the Lorentzian property of the quadratic

$$h_{2,G}(z) = \sum_{\{u,v\} \subseteq V} W_{uv} z_u z_v = \tfrac{1}{2} z^{\mathsf{T}} W z, \tag{2}$$

whose Hessian is W [1].

Our first theorem determines the full inertia for line graphs of subcubic graphs, whose maximum degree is at most three. Let H be connected with a positive even number of edges. Deleting any bridge leaves one side with an odd number of edges and one with an even number. Direct the bridge toward the odd side and replace its tail by a new leaf. Apply this to every bridge, using the directions determined in the original graph, and denote the nonempty components by $H_1, \ldots, H_\kappa$. A bridge is pendant if one endpoint has degree one.

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Theorem 1.1 (Subcubic line graphs)**.** *If H is connected, simple, subcubic, and has a positive even number of edges, then*

$$n_+\big(W(L(H))\big) = \kappa.$$

Thus $W(L(H))$ has one positive eigenvalue if and only if every bridge of H is pendant. Theorem 3.5 gives the negative index and nullity.

The proof separates the components H_i by a congruence and factors each remaining matrix using two-edge cuts. It also identifies κ with the number of components formed by edges of $L(H)$ that occur in perfect matchings, giving a linear-time algorithm for the positive index. A family on $4r$ vertices has inertia $(r, 2r-1, r+1)$ despite having least adjacency eigenvalue -2.

The second theorem treats strongly regular graphs.

Theorem 1.2 (Strongly regular graphs)**.** *Let G be a connected strongly regular graph of even order n with least adjacency eigenvalue -2. Then G has a perfect matching and*

$$\operatorname{In}(W(G)) = \begin{cases} (1,1,2), & G \cong C_4, \\ (1, n-1, 0), & G \ncong C_4. \end{cases}$$

For the lattice and triangular families, symmetry reduces the claim to two inequalities between deletion counts. Gaussian moments and switching prove both inequalities strictly. The Shrikhande and Chang graphs require no rank-three assumption: exact cofactor counts and a perturbation bound prove negativity on $\mathbf{1}^{\perp}$.

Matching polynomials and their deletion identities are classical [9, 7, 8]; Isserlis' formula supplies the Gaussian representation [10]. The enumeration for line graphs of bases with degrees two or three is due to Yan and Zhang [13]; see also [6]. We use its parity proof, including pendant vertices, to study the full deletion matrix rather than the total count. Pair deletion also appears in PM-compact graphs, where the question is uniqueness [14].

The strongly regular theorem uses Seidel's classification within the theory of graphs with least eigenvalue -2 [11, 5, 2, 3], and standard Chang switching models [4]. Its finite cases are computer-assisted; the accompanying code reconstructs the graphs and integer certificates. Section 3 proves the subcubic theorem, Sections 4–9 prove the strongly regular theorem, and Section 10 gives the remaining conjectures.

2. General representations

Let $\Omega_2(G)$ be the matchings exposing exactly two vertices. For $F \in \Omega_2(G)$, write $\partial F = \{u_F, v_F\}$. Let d_u count such matchings exposing u, put $D = \operatorname{diag}(d_u)$, and give T_+ and T_- the respective rows $e_{u_F} + e_{v_F}$ and $e_{u_F} - e_{v_F}$. The signs may be chosen arbitrarily.

Proposition 2.1. *One has*

$$T_+^{\mathsf{T}} T_+ = D + W, \qquad T_-^{\mathsf{T}} T_- = D - W, \qquad -D \preceq W \preceq D. \tag{3}$$

If every $d_u > 0$, then $P = D^{-1}W$ is a reversible Markov chain with stationary weights proportional to d_u. It is similar to $K = D^{-1/2} W D^{-1/2}$, which has the same inertia as W. Hence $n_+(W) = 1$ exactly when all eigenvalues of P, apart from one copy of the stationary eigenvalue 1, are nonpositive.

Proof. The Gram entries count exposed pairs with the indicated signs. The remaining claims follow from $W\mathbf{1} = D\mathbf{1}$, the congruence defining K, and $d_u P_{uv} = W_{uv} = d_v P_{vu}$. □

Proposition 2.2. *Let A be the adjacency matrix of G and s its least eigenvalue. If Z is centered Gaussian with covariance $A - sI$, then for every even subset $S \subseteq V(G)$,*

$$\operatorname{pm}(G[S]) = \mathbb{E} \prod_{v \in S} Z_v. \tag{4}$$

For $G = L(H)$, with H simple, independent standard Gaussians $(g_x)_{x\in V(H)}$ give

$$\operatorname{pm}(L(H)) = \mathbb{E} \prod_{xy\in E(H)} (g_x + g_y). \tag{5}$$

Deleting factors gives the cofactor formulas.

Proof. Since $A - sI \succeq 0$, the Gaussian exists. Isserlis' formula [10] sums covariance products over pairings; the nonzero terms are exactly the perfect matchings. No diagonal covariance occurs. For a line graph, the unsigned incidence matrix B satisfies $B^{\mathsf{T}}B = A(L(H)) + 2I$, the covariance of $Z_{xy} = g_x + g_y$. □

Corollary 2.3. *If W has one positive eigenvalue, then $h_{2,G}$ is Lorentzian and, for nonnegative x, y with $x^{\mathsf{T}}Wx, y^{\mathsf{T}}Wy > 0$,*

$$(x^{\mathsf{T}}Wy)^2 \geq (x^{\mathsf{T}}Wx)(y^{\mathsf{T}}Wy).$$

Proof. The form is negative semidefinite on the W-orthogonal complement of x. Apply it to $y - (x^{\mathsf{T}}Wy)/(x^{\mathsf{T}}Wx)x$. The Lorentzian interpretation is the quadratic case of [1]. □

3. Subcubic line graphs

All base graphs in this section are simple. In component counts, isolated vertices are included unless stated otherwise.

3.1. **Counting matchings and attaching graphs.**

Lemma 3.1. *If H has maximum degree at most three, then*

$$\operatorname{pm}(L(H)) = \begin{cases} 2^{|E(H)|-|V(H)|+c(H)}, & \text{every component has an even number of edges,} \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

where $c(H)$ includes isolated vertices. For distinct edges e, f,

$$W(L(H))_{ef} = \begin{cases} 2^{|E(H)|-2-|V(H)|+c(H-e-f)}, & \text{every component of } H-e-f \text{ has even size,} \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where size means number of edges.

Proof. A perfect matching of $L(H)$ is a P_3-decomposition of H, that is, a partition of its edges into paths of length two. Orient the edges toward their path centers. Every indegree is zero or two, so the orientation also determines the decomposition uniquely.

Relative to any initial orientation, encode reversals in $\mathbb{F}_2^{E(H)}$. The even-indegree constraints are incidence equations of rank $|V(H)| - c(H)$, consistent exactly when each component has an even number of edges. This proves (6); the deletion formula follows from $L(H)-e-f = L(H-e-f)$. □

Theorem 3.2 (Attachment at a vertex)**.** *Let A, B be disjoint graphs of even order with $M_A = \operatorname{pm}(A) > 0$ and $M_B = \operatorname{pm}(B) > 0$. Obtain G by joining a fixed $b \in V(B)$ to every vertex of $S \subseteq V(A)$. Put*

$$Q_A = \frac{W(A)}{M_A}, \quad Q_B = \frac{W(B)}{M_B}, \quad a = Q_A\mathbf{1}_S, \quad c = Q_B e_b.$$

Then $\operatorname{pm}(G) = M_A M_B$ and

$$Q_G := \frac{W(G)}{\operatorname{pm}(G)} = \begin{pmatrix} Q_A & ac^{\mathsf{T}} \\ ca^{\mathsf{T}} & Q_B \end{pmatrix}, \tag{8}$$

with the congruence

$$\begin{pmatrix} I & -ae_b^{\mathsf{T}} \\ 0 & I \end{pmatrix} Q_G \begin{pmatrix} I & 0 \\ -e_b a^{\mathsf{T}} & I \end{pmatrix} = Q_A \oplus Q_B. \tag{9}$$

In particular, $\operatorname{In}(W(G)) = \operatorname{In}(W(A)) + \operatorname{In}(W(B))$.

Proof. A matching uses at most one added edge. Parity prevents a perfect matching from using exactly one, so the counts multiply. The same argument after two deletions within A or B gives the diagonal blocks. Deleting $u \in A$ and $v \in B$ forces one added edge, and hence

$$W(G)_{uv} = \sum_{s \in S} W(A)_{us} W(B)_{vb}.$$

Zero cofactor diagonals cover $u = s$ and $v = b$. Division by $M_A M_B$ gives (8). Since $Q_B e_b = c$ and $e_b^{\mathsf{T}} Q_B e_b = 0$, multiplication proves (9). Its triangular matrices are invertible, so inertia is additive. □

3.2. **Separating at bridges.**

Lemma 3.3. *Let H satisfy Theorem 1.1, and let $H_1, \dots, H_\kappa$ be the nonempty components after detaching bridges as in the introduction. Each H_i has a positive even number of edges and only pendant bridges. With $M_i = \mathrm{pm}(L(H_i))$,*

$$\frac{W(L(H))}{\mathrm{pm}(L(H))} \;\cong\; \bigoplus_{i=1}^{\kappa} \frac{W(L(H_i))}{M_i}, \qquad \mathrm{pm}(L(H)) = \prod_i M_i, \tag{10}$$

where $\cong$ denotes real congruence.

Proof. Lemma 3.1 supplies an orientation with even indegrees. Summing indegrees over one side of a bridge shows that the bridge enters that side exactly when its internal edge count is odd. Detaching its tail preserves the old indegrees and gives the new leaf indegree zero. Thus every resulting component has an even number of edges. Original nonbridges remain on cycles, and detached bridges are pendant.

Choose a nonpendant bridge xy whose even side H_A contains x. Let H_B consist of the odd side and xy, with x replaced by a leaf. Both graphs have positive even size. To recover $L(H)$ from $L(H_A) \sqcup L(H_B)$, join the vertex xy to the edges of H_A incident with x. Theorem 3.2 gives one step of (10).

Repeat until only pendant bridges remain. Each removed side has even size, so these steps do not change the parity direction of any remaining bridge. Detaching a pendant bridge merely replaces its leaf. The final components are therefore the same H_i as in simultaneous detachment, up to isolated vertices and leaf labels. □

3.3. **Factorization and inertia.** Let K be connected and subcubic, with a positive even number of edges and only pendant bridges. Write P for its pendant edges. For nonbridge edges, define $e \sim f$ when $e = f$ or $\{e, f\}$ is a two-edge cut. This is an equivalence relation: two nonbridge edges form a cut exactly when their nonzero coordinate functionals on the binary cycle space agree. Indeed, if no cycle contains e without f, then e is a bridge in $K - f$; conversely, every cycle crosses a cut evenly.

Adjoin the pendant edges as singleton classes. For each nonbridge class S, choose a cycle containing one, hence all, of its edges, and orient it cyclically. Fix an orientation ω of K with even indegrees. Give $e \in S$ sign $\sigma_e = 1$ when ω agrees with the cycle and -1 otherwise. Let u_S have these signs on S and zero elsewhere; pendant singletons have sign 1. Define

$$D_{ee} = \begin{cases} 2, & e \in P, \\ 1, & e \notin P. \end{cases} \tag{11}$$

Theorem 3.4. *With $M = \mathrm{pm}(L(K))$,*

$$\frac{4}{M} W(L(K)) = D \left(J - \sum_S u_S u_S^{\mathsf{T}} \right) D. \tag{12}$$

The outer products are independent of the chosen cycles and orientation. For distinct nonbridge edges,

$$W_{ef} = \begin{cases} M/4, & e, f \text{ belong to different classes,} \\ 0, & e, f \text{ belong to one class and have equal signs,} \\ M/2, & e, f \text{ belong to one class and have opposite signs.} \end{cases}$$

The cofactor is $M/2$ between a pendant edge and a nonbridge edge, and M between two pendant edges.

Proof. Deleting nonbridge edges in different classes leaves K connected, giving $M/4$ by Lemma 3.1. Deleting two in the same class leaves two components. The chosen cycle crosses the cut in opposite directions. Thus the two edges point into either component an odd number of times exactly when their signs agree. Since ω has even indegrees, this is also the parity of that component's internal edge count. Equal signs therefore give two odd components and cofactor zero; opposite signs give two even components and $M/2$.

Deleting a pendant edge and a nonbridge edge leaves an isolated leaf and a connected remainder of even size, giving $M/2$. Deleting two pendant edges leaves two isolated leaves and a connected remainder, giving M. For a two-edge path, the remainder is a single vertex. These cases prove (12), including its zero diagonal.

The products $\sigma_e\sigma_f$ are determined by the parities of the cut components. Changing the choices can only reverse all signs of a class, preserving its outer product. □

Theorem 3.5. *For K as above, with m edges and t classes,*

$$\operatorname{In}(W(L(K))) = \begin{cases} (1, t, m-t-1), & \text{some } u_S \text{ contains both signs,} \\ (1, t-1, m-t), & \text{otherwise.} \end{cases} \tag{13}$$

For any connected subcubic H of positive even size, sum these triples over $H_1, \dots, H_\kappa$. In particular,

$$n_+(W(L(H))) = \kappa, \tag{14}$$

and this index is one exactly when every bridge of H is pendant.

Proof. The vectors u_S have disjoint nonempty supports. Let U have these vectors as columns. The factorization reduces inertia to that of $B = J - UU^{\mathsf{T}}$.

If some u_S contains both signs, then $\mathbf{1} \notin \operatorname{im} U$, so $[\mathbf{1}\ U]$ has full column rank. The identity

$$B = [\mathbf{1}\ U] \operatorname{diag}(1, -I_t)[\mathbf{1}\ U]^{\mathsf{T}}$$

gives inertia $(1, t, m-t-1)$ by extending its columns to a basis.

Otherwise, reverse all signs within a class to make all entries $+1$. Then B is the adjacency matrix of a complete multipartite graph with t parts. Its kernel consists of vectors with sum zero in each part; on the space of vectors constant on each part, its form is congruent to $J_t - I_t$. Hence its inertia is $(1, t-1, m-t)$. Here $t \geq 2$, since a perfect matching of $L(K)$ supplies a positive cofactor.

Lemma 3.3 makes inertia additive. If every bridge is pendant, detachment leaves one nonempty component. A nonpendant bridge separates a positive even side from an odd side which, with the bridge attached, also has positive even size. Both contribute at least one component, so $\kappa \geq 2$. □

Corollary 3.6. *Under Theorem 1.1, let $G = L(H)$ and let G^{allow} contain exactly the edges of G occurring in some perfect matching. Then*

$$n_+(W(G)) = \#\{\text{connected components of } G^{\text{allow}}\}.$$

Proof. No attachment edge in Theorem 3.2 occurs in a perfect matching. Thus it suffices to prove connectivity within $L(K)$ when K has only pendant bridges.

At a vertex incident with two nonbridge edges and no others, deleting those edges isolates the vertex and leaves an even connected remainder, so the corresponding adjacency is allowed. At a

vertex incident with three nonbridge edges, no two form a cut, since the third would then be a bridge. All three adjacencies are therefore allowed. With two nonbridge edges and one pendant edge, the pendant edge has an allowed adjacency to each of the others. Hence every local clique in the cyclic part of $L(K)$ has a connected allowed subgraph, proving connectivity. Without cycles, K is an even star of maximum degree three, hence a two-edge path, and $L(K) = K_2$. □

Corollary 3.7. *The positive index of $W(L(H))$ is computable in $O(|V(H)|+|E(H)|)$ time without counting matchings or constructing W.*

Proof. Depth-first search finds the bridges and the degree sum in each subtree S separated by a bridge. Its internal edge count is $(\sum_{v\in S} \deg_H(v) - 1)/2$, whose parity determines the bridge direction. Detach the tails and count nonempty components by another graph search. □

Corollary 3.8. *Every connected bridgeless subcubic graph of positive even size has a Lorentzian polynomial $h_{2,L(H)}$. If H is three-edge-connected and cubic on N vertices, with an even number of edges, then*

$$W(L(H)) = 2^{N/2-1}(J - I). \tag{15}$$

Proof. The first claim follows from Theorem 3.5. In the second case, every class is a singleton and $D = I$. Since $M = 2^{3N/2-N+1} = 2^{N/2+1}$, (12) gives (15). □

3.4. **Unbounded positive index and four monomers.**

Theorem 3.9. *For every integer $r \geq 1$, there is a connected simple subcubic graph H_r with $4r$ edges such that $L(H_r)$ has least adjacency eigenvalue -2 and*

$$\operatorname{In}(W(L(H_r))) = (r, 2r - 1, r + 1). \tag{16}$$

Take a 4-cycle followed by $r-1$ disjoint triangles, joined consecutively by bridges at distinct vertices of each internal cycle.

Proof. The base has $4r$ edges and $3r + 1$ vertices. Every bridge points away from the first cycle, whose side has even size. Detachment leaves the 4-cycle and $r - 1$ triangles, each with a pendant edge.

The 4-cycle has one class containing both signs and inertia $(1, 1, 2)$. A triangle with a pendant edge has two classes. Its triangle class contains both signs: deleting the two edges at the attachment leaves an odd component, whereas another pair isolates a vertex. Its inertia is $(1, 2, 1)$, giving (16) by addition.

The incidence identity $A(L(H_r)) + 2I = B^{\mathsf{T}}B \succeq 0$ gives the spectral lower bound. For $r \geq 2$, $|E(H_r)| > |V(H_r)|$, so the matrix is singular. For $r = 1$, the line graph is C_4. □

For even j, the polynomial for j monomers is

$$h_{j,G}(z) = \sum_{\substack{S\subseteq V(G)\\ |S|=j}} \operatorname{pm}(G - S) \prod_{v\in S} z_v. \tag{17}$$

Theorem 3.10. *There is a three-edge-connected cubic graph H on eight vertices for which $h_{2,L(H)}$ is Lorentzian but $h_{4,L(H)}$ is not.*

Proof. Take vertices $1, \ldots, 8$ and edges

$$\{12, 34, 45, 35, 14, 25, 67, 78, 68, 16, 27, 38\}. \tag{18}$$

This replaces one vertex of a triangular prism by a triangle, attaching the three incident edges to distinct new vertices. The graph is cubic. A cut splitting the new triangle crosses two of its edges and at least one more, since the prism minus the replaced vertex is connected. A cut not splitting the triangle contracts to a cut in the prism. Thus H is three-edge-connected, and (15) gives $W(L(H)) = 8(J - I)$.

Delete $e = 16$ and $f = 27$. The edge 38 becomes a bridge, with triangle 678 on its odd side and the six-edge graph $\{12, 34, 45, 35, 14, 25\}$ on its even side. Detachment leaves a triangle with a pendant edge, of inertia $(1, 2, 1)$, and a theta graph with path lengths $1, 2, 3$. The latter has three classes, one containing both signs, and inertia $(1, 3, 2)$. Hence

$$\mathrm{In}\big(W(L(H) - e - f)\big) = (2, 5, 3).$$

Since $\partial_e \partial_f h_{4,L(H)} = h_{2,L(H)-e-f}$, this derivative has two positive Hessian eigenvalues. Derivatives of Lorentzian polynomials are Lorentzian [1], so $h_{4,L(H)}$ cannot be Lorentzian. □

4. Rank-three graphs

Assume that G is connected and noncomplete, has even order n and a perfect matching, and that $\mathrm{Aut}(G)$ has three orbits on ordered vertex pairs. Such a graph is strongly regular. Write $k > r > s$ for its adjacency eigenvalues and $\ell = n - 1 - k$ for its number of nonneighbors per vertex.

Let p_j count the j-edge matchings, so the matching polynomial is

$$\mu_G(x) = \sum_{j=0}^{n/2} (-1)^j p_j x^{n-2j}. \tag{19}$$

Put $M = p_{n/2}$ and $p = p_{n/2-1}$. Symmetry makes the deletion counts constant on edges and nonedges; denote them by a and b, respectively.

Theorem 4.1. *One has*

$$W = aA + b(J - I - A), \qquad a = M/k > 0. \tag{20}$$

Moreover,

$$p = \frac{n}{2}(ka + \ell b), \qquad W\mathbf{1} = \frac{2p}{n}\mathbf{1}. \tag{21}$$

Writing $q = b/a$, the eigenvalues of W are

$$\phi = a(k + \ell q) = 2p/n > 0, \qquad \theta_t = a\big(t - (t+1)q\big) \quad (t = r, s), \tag{22}$$

with the corresponding adjacency multiplicities. Thus

$$n_+(W) = 1 \quad \Longleftrightarrow \quad \frac{r}{r+1} \le \frac{b}{a} \le \frac{s}{s+1}. \tag{23}$$

Proof. Transitivity on edges and nonedges gives (20). Exposing the partner of a fixed vertex gives $M = ka$, and counting the exposed pair gives $p = n(ka + \ell b)/2$. On $\mathbf{1}^\perp$, the matrix J vanishes, so $W = (a - b)A + b(J - I)$ has eigenvalues $(a - b)t - b$. Since $r \ge 0$ and $s < -1$, their nonpositivity is equivalent to (23). □

The comparison $a \ge b$ implies the upper bound in (23), since $s/(s + 1) > 1$, but not the lower bound.

Put $\rho = p/M$ and let α be the probability that the two vertices exposed by a uniform $F \in \Omega_2(G)$ are adjacent. Counting those pairs gives

$$q = \frac{2k}{n\ell}\left(\rho - \frac{n}{2}\right), \qquad \alpha = \frac{nM}{2p} = \frac{n}{2\rho} = \frac{k}{k + \ell q}. \tag{24}$$

Corollary 4.2. *The condition $n_+(W) = 1$ is equivalent to either of the intervals*

$$\frac{n}{2} + \frac{n\ell r}{2k(r+1)} \le \rho \le \frac{n}{2} + \frac{n\ell s}{2k(s+1)}, \tag{25}$$

$$\frac{k(s+1)}{k + s(n-1)} \le \alpha \le \frac{k(r+1)}{k + r(n-1)}. \tag{26}$$

Proof. Substitute (24) into (23). □

4.1. **Augmenting paths of length three.** For $F \in \Omega_2(G)$ with exposed vertices u, v, let $c_3(F)$ count the ordered matching edges (x, y) with $ux, vy \in E$. Each gives an augmenting path u, x, y, v. Interchanging u, v leaves the count unchanged.

Theorem 4.3. *For uniform $F \in \Omega_2(G)$,*

$$\mathbb{E}[c_3(F)] = \frac{n(k-1)}{2\rho} = (k-1)\alpha. \tag{27}$$

Thus (23) *is also equivalent to*

$$(k-1)\frac{k(s+1)}{k+s(n-1)} \le \mathbb{E}[c_3(F)] \le (k-1)\frac{k(r+1)}{k+r(n-1)}. \tag{28}$$

Proof. Given a perfect matching P and $xy \in E \setminus P$, let u, v be the partners of x, y. Replacing ux, vy by xy produces F and an augmenting path counted by c_3. The operation is reversible, so

$$\sum_F c_3(F) = M\left(|E| - \tfrac{n}{2}\right) = \frac{Mn(k-1)}{2}.$$

Divide by $p = M\rho$ and use (24). □

For exposed u, v, partition the remaining vertices into

$$C = N(u) \cap N(v), \qquad X = N(u) \setminus (C \cup \{v\}),$$
$$Y = N(v) \setminus (C \cup \{u\}), \qquad \mathcal{R} = V \setminus (\{u, v\} \cup C \cup X \cup Y).$$

Let $m_F(S, T)$ count matching edges joining S to T, including edges within S when $S = T$.

Lemma 4.4. *For every $F \in \Omega_2(G)$,*

$$c_3(F) = |C| + m_F(X, Y) - m_F(C, \mathcal{R}). \tag{29}$$

Proof. An edge within C contributes twice to c_3, and edges between C, X, between C, Y, or between X, Y contribute once. Hence

$$c_3 = 2m_F(C, C) + m_F(C, X) + m_F(C, Y) + m_F(X, Y),$$
$$|C| = 2m_F(C, C) + m_F(C, X) + m_F(C, Y) + m_F(C, \mathcal{R}).$$

Subtract the second equality, which counts the matched vertices of C. □

5. Complete multipartite graphs

Let $P(t_1, \dots, t_m)$ count perfect matchings in the complete multipartite graph with the indicated part sizes, and set $P = 0$ for odd total order.

Lemma 5.1. *For fixed remaining part sizes π and $1 \le \sigma \le \tau$,*

$$P(\sigma, \tau, \pi) \ge P(\sigma - 1, \tau + 1, \pi). \tag{30}$$

Proof. Induct on total order, the cases at most two being immediate. Expose a vertex in the first part on the left and in the second part on the right. Contributions from partners in π cancel, leaving

$$P(\sigma, \tau, \pi) - P(\sigma - 1, \tau + 1, \pi) = \tau U - (\sigma - 1)U', \tag{31}$$

where $U = P(\sigma - 1, \tau - 1, \pi)$ and $U' = P(\sigma - 2, \tau, \pi)$. For $\sigma = 1$, the second term is absent. Otherwise, induction gives $U \ge U' \ge 0$, so the difference is nonnegative. □

Theorem 5.2. *If $G = K_{m \times t}$ has $m \ge 2$ equal parts of size $t \ge 2$ and mt is even, then $W(G)$ has one positive eigenvalue.*

Proof. List the parts consecutively and match each vertex to the one halfway around the cyclic list; these pairs lie in different parts. Deleting adjacent or nonadjacent vertices gives, respectively,

$$a = P(t-1, t-1, t, \dots, t), \qquad b = P(t-2, t, t, \dots, t).$$

Lemma 5.1 gives $a \geq b$. The adjacency eigenvalues are $(m-1)t, 0, -t$, so the required interval is $0 \leq b/a \leq t/(t-1)$, which holds. □

This includes complete bipartite and cocktail-party graphs. For $t = 1$, the graph is K_m with m even, and $W = (m-3)!!(J-I)$ has eigenvalues $(m-3)!!(m-1)$ and $-(m-3)!!$, the latter with multiplicity $m-1$.

6. Lattice graphs

The lattice graph $R_m = K_m \square K_m = L(K_{m,m})$ has vertex set $[m] \times [m]$, with adjacency along rows and columns. Assume $m \geq 4$ is even; pairing within rows gives a perfect matching. The case $m = 2$ is C_4. Its parameters are

$$n = m^2, \quad k = 2(m-1), \quad \ell = (m-1)^2, \quad r = m-2, \quad s = -2, \tag{32}$$

so (23) becomes

$$\frac{m-2}{m-1} \leq \frac{b}{a} \leq 2. \tag{33}$$

6.1. **A Gaussian identity.** In (5), write $X_{ij} = \xi_i + \zeta_j$, with independent standard Gaussians ξ_i, ζ_j. For an independent standard Gaussian ξ and $A \subseteq [m]$, put

$$F(\zeta) = \mathbb{E}_\xi \prod_{j=1}^{m} (\xi + \zeta_j), \qquad F_A = \partial_A F. \tag{34}$$

Since F is multilinear, F_A is the same expectation with the factors indexed by A omitted. Write $F_j = \partial_j F$ and $F_{1,2} = \partial_1 \partial_2 F$.

Theorem 6.1. *For R_m,*

$$M = \mathbb{E}[F^m], \qquad a = \mathbb{E}[F_1^2 F^{m-2}], \qquad b = \mathbb{E}[F_1 F_2 F^{m-2}], \tag{35}$$

and

$$a - b = \frac{1}{2}\mathbb{E}\left[(\zeta_1 - \zeta_2)^2 F_{1,2}^2 F^{m-2}\right] \geq 0. \tag{36}$$

Proof. Condition on ζ in (5) and integrate the row variables to obtain $M = \mathbb{E}[F^m]$. Deleting $(1,1),(2,1)$ replaces two row factors by F_1; deleting $(1,1),(2,2)$ replaces them by F_1, F_2. These give a, b.

Symmetry in ζ_1, ζ_2 gives $a - b = \frac{1}{2}\mathbb{E}[(F_1 - F_2)^2 F^{m-2}]$. Now $F_1 - F_2 = (\zeta_2 - \zeta_1) F_{1,2}$, and $F^{m-2} \geq 0$ because m is even. □

6.2. **Switching and variance.** For a perfect matching P of R_m, let $h_i(P)$ count its horizontal edges in row i and $v_j(P)$ its vertical edges in column j. Symmetry gives

$$\mathbb{E}h_i = \mathbb{E}v_j = m/4. \tag{37}$$

Let $\mathcal{B}_m$ count matchings exposing a nonadjacent pair. Equivalently, it counts pairings of all cells with exactly one pair not sharing a row or column. Then

$$\mathcal{B}_m = \frac{m^2(m-1)^2}{2}\, b. \tag{38}$$

Lemma 6.2. *For the set $\mathcal{P}_m$ of perfect matchings of R_m,*

$$\sum_{P\in\mathcal{P}_m}\left[\sum_i 2h_i(P)(m-2h_i(P))+\sum_j 2v_j(P)(m-2v_j(P))\right]=2\mathcal{B}_m. \tag{39}$$

Proof. In row i, choose a horizontal edge $\{x,y\}$, an endpoint x, and a vertically matched cell z, with partner w. Replace $\{x,y\},\{z,w\}$ by $\{y,z\},\{x,w\}$. The latter pair is the unique nonedge. There are $2h_i(m-2h_i)$ choices; columns give the second sum.

Conversely, the unique nonedge $\{x,w\}$ determines a rectangle. Either of its other corners gives a reversal, and these are the only two preimages. □

Theorem 6.3. *For the horizontal count h_1 in a uniform perfect matching,*

$$\frac{b}{M}=\frac{m^2/2-8\operatorname{Var}(h_1)}{m(m-1)^2}. \tag{40}$$

The following are equivalent: $n_+(W(R_m))=1$, $(m-1)b\ge(m-2)a$, $\operatorname{Var}(h_1)\le m/8$, and $\rho\ge m^3/4$.

Proof. All row and column counts have the same distribution, and $\mathbb{E}[2h_i(m-2h_i)]=m^2/4-4\operatorname{Var}(h_i)$. Thus (39) gives

$$M\left(\frac{m^3}{2}-8m\operatorname{Var}(h_1)\right)=m^2(m-1)^2b.$$

Use $M=2(m-1)a$, (33), $b\le a$, and Corollary 4.2 for the equivalences. □

6.3. **A path identity.** Put $\Delta=\sum_{i=1}^m\partial_i$ and, for $|A|=k$, define

$$T_k=\mathbb{E}[F_A^2F^{m-2}]. \tag{41}$$

Symmetry makes this independent of A. By (5), T_k counts perfect matchings after deleting a $2\times k$ rectangle of cells. Transposing gives P_3-decompositions of $K_{m,m}$ with a $k\times 2$ rectangle of edges removed.

Lemma 6.4. *One has*

$$(\Delta F)^2-F\Delta^2F=\sum_{\varnothing\neq A\subseteq[m]}(-1)^{|A|+1}|A|!\,F_A^2, \tag{42}$$

and consequently

$$m\big((m-1)b-(m-2)a\big)=\sum_{k=1}^{m}(-1)^{k+1}(m)_kT_k, \tag{43}$$

where $(m)_k=m(m-1)\cdots(m-k+1)$.

Proof. The positive-coefficient form of the matching-polynomial path identity [8] is

$$F_iF_j-FF_{i,j}=\sum_{P:i\rightsquigarrow j}(-1)^{|E(P)|}F_{V(P)}^2\qquad(i\neq j).$$

Sum over ordered $i\neq j$ and add $\sum_i F_i^2$. A set A of size $k\ge 2$ supports $k!$ directed Hamilton paths; singleton sets occur once. This gives (42).

Multiply by F^{m-2} and take expectations. The two left-hand terms give $ma+m(m-1)b$ and $m(m-1)a$, respectively. Each set of size k contributes T_k, with $k!\binom{m}{k}=(m)_k$. □

Pair consecutive terms in (43). It remains to prove

$$T_k\ge(m-k)T_{k+1}\qquad(k\text{ odd}). \tag{44}$$

6.4. **Conditioning on two columns.** Fix odd $k \in \{1, \dots, m-1\}$ and put $\nu = m-k$, $\mu = m-2$. In the graph defining T_k, label the rows so that rows $1, \dots, \nu$ meet columns $1, 2$, while the other k rows do not. An orientation with even indegrees represents

$$\prod_v (d_\omega^-(v) - 1)!! \tag{45}$$

path decompositions, by pairing incoming edges at each vertex; $(-1)!! = 1$.

Condition on the orientations in columns $3, \dots, m$. Let y_i count edges directed toward row i. For $i > \nu$, y_i must be even. Among $i \le \nu$, let $\mathcal{O}$ contain those with y_i odd and let $\mathcal{E}$ be the rest. Write $o = |\mathcal{O}|$ and $e = |\mathcal{E}| = \nu - o$. Column parity forces o even, so e is odd. Set $\lambda_i = y_i + 1$ for $i \in \mathcal{E}$. If c_j counts edges directed toward rows in column j, the common weight is

$$w(\omega) = \prod_{j=3}^{m} (m - c_j - 1)!! \prod_{i>\nu} (y_i - 1)!! \prod_{i \in \mathcal{E}} (y_i - 1)!! \prod_{i \in \mathcal{O}} y_i!!. \tag{46}$$

For independent standard Gaussians γ_1, γ_2, put

$$K_o(t) = \mathbb{E}[\gamma_1^t \gamma_2^t (\gamma_1 + \gamma_2)^o]. \tag{47}$$

With σ_a denoting the elementary symmetric polynomial, the remaining total weight is

$$\mathcal{Z}_o(\lambda) = \sum_{a=0}^{e} \sigma_a(\lambda) K_o(e - a), \tag{48}$$

$$\mathcal{N}_o(\lambda) = \sum_{a=0}^{e-1} (e - a) \sigma_a(\lambda) K_o(e - a - 1). \tag{49}$$

Here $\mathcal{N}_o$ counts the same decompositions with one marked path joining the two distinguished edges in a row. To see these formulas, an even y_i requires zero or two of these edges to point toward row i, with relative weights $1, \lambda_i$; an odd y_i requires exactly one. Gaussian moments count the pairings in columns $1, 2$. When both edges point toward a row, they are paired together in a fraction $1/\lambda_i$ of its pairings, giving (49).

Deleting the two edges of a marked path gives a decomposition counted by T_{k+1}. There are ν possible rows, so

$$T_k - (m-k) T_{k+1} = \sum_\omega w(\omega) \big(\mathcal{Z}_{o(\omega)} - \mathcal{N}_{o(\omega)} \big). \tag{50}$$

Lemma 6.5. *For even $o \ge 2$ and $t \ge 1$,*

$$K_o(t) \ge t K_o(t-1). \tag{51}$$

Hence $\mathcal{Z}_o - \mathcal{N}_o \ge 0$ for $o \ge 2$.

Proof. Write $o = 2j$ and set $U = (\gamma_1 + \gamma_2)/\sqrt{2}$, $U' = (\gamma_1 - \gamma_2)/\sqrt{2}$. These are independent standard Gaussians, and

$$\begin{aligned} \sum_{t \ge 0} \frac{K_{2j}(t)}{t!} z^t &= \mathbb{E}[(\gamma_1 + \gamma_2)^{2j} e^{z \gamma_1 \gamma_2}] \\ &= 2^j \mathbb{E}[U^{2j} e^{z U^2/2}] \mathbb{E}[e^{-z U'^2/2}] \\ &= 2^j (2j-1)!! (1-z)^{-j} (1-z^2)^{-1/2}. \end{aligned}$$

After multiplication by $1 - z$, every coefficient remains nonnegative because $j \ge 1$. This proves (51); apply it termwise to (48)–(49). □

Only the terms with $o = 0$ can be negative. Put $\mathcal{D}_o = \mathcal{Z}_o - \mathcal{N}_o$, and write $\lambda_{\widehat{i,j}}$ for λ with coordinates i, j removed.

Lemma 6.6. *For odd ν and $\lambda = (\lambda_1, \dots, \lambda_\nu)$ with $\lambda_i \geq 1$,*

$$\mathcal{D}_0(\lambda) + \frac{1}{3} \sum_{i<j} \mathcal{D}_2(\lambda_{\widehat{i,j}}) \geq 0. \tag{52}$$

Proof. Let $Q(z) = \prod_{h=1}^{\nu}(z + \lambda_h)$ and $Q_{i,j} = Q/((z + \lambda_i)(z + \lambda_j))$. Equations (48)–(49) give

$$\mathcal{D}_0(\lambda) = \mathbb{E}[(Q - Q')(\gamma_1\gamma_2)],$$
$$\mathcal{D}_2(\lambda_{\widehat{i,j}}) = \mathbb{E}[(\gamma_1 + \gamma_2)^2(Q_{i,j} - Q'_{i,j})(\gamma_1\gamma_2)].$$

Since $\sum_{i<j} Q_{i,j} = Q''/2$, the left side of (52) is

$$\mathcal{L}(Q) = \mathbb{E}\left[Q - Q' + \tfrac{1}{6}(\gamma_1 + \gamma_2)^2(Q'' - Q''')\right]_{z=\gamma_1\gamma_2}. \tag{53}$$

Writing $\lambda_h = 1 + y_h$ expands Q as

$$Q(z) = \sum_{J \subseteq [\nu]} \left(\prod_{h \in J} y_h \right) (z + 1)^{\nu - |J|}.$$

It therefore suffices to show $\mathcal{L}((z + 1)^t) \geq 0$ for $t \geq 0$. Put $\Psi = 1 + \gamma_1\gamma_2$. Integration by parts in $U = (\gamma_1 + \gamma_2)/\sqrt{2}$ yields

$$\mathcal{L}((z + 1)^t) = \mathbb{E}\left[\Psi^t - t\Psi^{t-1} + \frac{t(t-1)}{3}\Psi^{t-2}\right].$$

For even $t \geq 4$, factor the integrand as $\Psi^{t-2}(\Psi^2 - t\Psi + t(t-1)/3)$; the quadratic has discriminant $t(4-t)/3 \leq 0$. The cases $t = 0, 1, 2$ are immediate. For odd t, symmetry of $\Gamma = \gamma_1\gamma_2$ reduces the expectation to its even part,

$$\sum_{j \geq 0} \binom{t}{2j} \frac{(t - 2j - 1)(t - 2j - 3)}{3} \Gamma^{2j},$$

whose coefficients are nonnegative. □

Lemma 6.7. *The combined contribution of $o = 0$ and $o = 2$ in (50) is nonnegative.*

Proof. Let Θ_0, Θ_2 be the conditioned orientations with $o = 0, 2$. For $\omega \in \Theta_2$, let i, j be its two odd rows. Reversing the edges in rows i, j of any column $c \geq 3$ gives $\tau_c\omega \in \Theta_0$. There are $\mu = m - 2$ such orientations.

Let d_c count edges directed toward rows in column c. The weight ratios are

$$\frac{w(\tau_c\omega)}{w(\omega)} = \begin{cases} 1/(m - d_c - 1), & 00, \\ 1/y_i, & 10, \\ 1/y_j, & 01, \\ (m - d_c + 1)/(y_i y_j), & 11, \end{cases}$$

where each bit indicates whether the edge points toward its row. The first three ratios are at most one. If κ columns have type 11, then $\kappa \leq y_i, y_j$ and $d_c \geq 2$, so their total ratio is at most $m - 1$. Therefore

$$\sum_{c=3}^{m} w(\tau_c\omega) \leq (2m - 3)w(\omega) \leq 3\mu w(\omega). \tag{54}$$

Distribute the nonnegative contribution $w(\omega)\mathcal{D}_2(\omega)$ among these orientations in proportion to their weights. Each $\tau_c\omega$ receives at least $w(\tau_c\omega)\mathcal{D}_2(\omega)/(3\mu)$.

Fix $\omega_0 \in \Theta_0$ with parameter vector λ. Reversing any pair of rows $i < j \le \nu$ in any column $c \ge 3$ gives a distinct preimage in Θ_2, with parameter vector $\lambda_{\widehat{i,j}}$ on the remaining even rows. Thus ω_0 receives at least

$$\frac{w(\omega_0)}{3} \sum_{i<j} \mathcal{D}_2(\lambda_{\widehat{i,j}}).$$

By (52), this covers any negative contribution $w(\omega_0)\mathcal{D}_0(\lambda)$. No contribution from Θ_2 is used twice. □

Theorem 6.8 (Lattice graphs)**.** *For every even $m \ge 4$,*

$$\frac{m-2}{m-1} < \frac{b}{a} < 1, \qquad \mathrm{In}(W(R_m)) = (1, m^2-1, 0).$$

Proof. Lemmas 6.5 and 6.7 make (50) nonnegative, proving (44). Pairing terms in (43) gives $(m-1)b \ge (m-2)a$.

For strictness, take $k = m-1$, so only row 1 meets columns $1, 2$. Then $o = 0$ and $\mathcal{D}_0(\lambda) = \lambda - 1 = y_1$. In each of two chosen columns among $3, \dots, m$, direct the edges in rows $1, 2$ toward their rows. Direct every other edge in columns $3, \dots, m$ toward its column. The two rows have indegree two, and every column indegree is even. This contributes positive weight with $\mathcal{D}_0 = 2$, so $T_{m-1} > T_m$. The last paired term in (43) is positive, and the others are nonnegative.

Finally, the integrand in (36) is a nonzero nonnegative polynomial, so $a > b$. Both nonprincipal eigenvalues in (22) are strictly negative. □

7. Counting path centers

Let H be connected and d-regular, with an even number of edges. For a P_3-decomposition $\mathcal{P}$, let $C_v(\mathcal{P})$ count paths centered at v. Choose $\mathcal{P}$ uniformly and $U \in V(H)$ uniformly and independently. Then $C = C_U(\mathcal{P})$ satisfies

$$\mathbb{E}C = d/4, \tag{55}$$

since each decomposition has $|E(H)|/2$ paths.

For $d > 2$, choose instead a uniform path of $\mathcal{P}$ and then a uniform edge among the other $d-2$ edges at its center. Let $\bar{\eta}_H$ be the probability that this edge also belongs to a path centered there.

Theorem 7.1. *For connected d-regular H of even size with $d > 2$,*

$$\mathrm{Var}(C) = \frac{d}{16}\big(4 - d + 2(d-2)\bar{\eta}_H\big). \tag{56}$$

In particular, $\mathrm{Var}(C) \le d/8$ *if and only if* $\bar{\eta}_H \le 1/2$.

Proof. The center of a uniform path is sampled with weight C_v. Of its $d-2$ remaining incident edges, $2C_v - 2$ are centered there. Thus

$$\bar{\eta}_H = \frac{2\mathbb{E}[C(C-1)]}{(d-2)\mathbb{E}C}.$$

Substitute $\mathbb{E}C = d/4$ and $\mathbb{E}[C(C-1)] = \mathrm{Var}(C) + (\mathbb{E}C)^2 - \mathbb{E}C$. □

Suppose also that H is vertex-transitive, the stabilizer of v is transitive on ordered triples of distinct incident edges, and the decomposition measure is symmetric under exchanging the endpoints of an edge. For such edges e, f, g, put

$$\eta_H = \mathbb{P}\{g \text{ belongs to a path centered at } v \mid e, f \text{ are paired at } v\}. \tag{57}$$

Symmetry makes this independent of the triple and equal to $\bar{\eta}_H$.

Corollary 7.2. *Under these symmetry assumptions,*

$$\operatorname{Var}(C_v) = \frac{d}{16}\big(4 - d + 2(d-2)\eta_H\big), \tag{58}$$

so $\operatorname{Var}(C_v) \le d/8$ *if and only if* $\eta_H \le 1/2$.

For $K_{m,m}$, C_v is a row or column count in R_m, and Theorem 6.8 gives this bound. For cubic bases, $C_v \in \{0,1\}$, so $\bar{\eta}_H = 0$. We next treat K_m.

8. Triangular graphs

Let $T(m) = L(K_m)$, with $m \ge 4$ and $m \equiv 0, 1 \pmod 4$. Its order is even, and it has a perfect matching by Sumner's theorem for connected claw-free graphs [12]. Its parameters are

$$n = \binom{m}{2}, \quad k = 2(m-2), \quad \ell = \frac{(m-2)(m-3)}{2}, \quad r = m-4, \quad s = -2. \tag{59}$$

Perfect matchings of $T(m)$ are P_3-decompositions of K_m. The counts a, b correspond to deleting two incident or two disjoint base edges, respectively. We must prove

$$(m-3)b \ge (m-4)a, \qquad b \le 2a. \tag{60}$$

8.1. The upper bound.

Theorem 8.1. *For every admissible* m*,*

$$b \le 2a, \qquad \textit{equivalently} \qquad \alpha \ge \frac{2}{m-1}, \tag{61}$$

where α *is the probability that the exposed base edges meet.*

Proof. Apply Lemma 4.4. For disjoint exposed edges $12, 34$, the common neighbors are $C = \{13, 14, 23, 24\}$. Every edge in $\mathcal{R}$ avoids $\{1,2,3,4\}$, so none is adjacent to C and $c_3(F) \ge 4$.

For incident exposed edges $12, 13$, one has $C = \{23\} \cup \{1j : 4 \le j \le m\}$. The vertex 23 has no neighbor in $\mathcal{R}$, so at most $|C| - 1$ vertices of C can be matched there. Hence $c_3(F) \ge 1$.

Averaging and using (27) gives

$$(2m-5)\alpha = \mathbb{E}[c_3(F)] \ge \alpha + 4(1-\alpha).$$

Thus $\alpha \ge 2/(m-1)$, or $b/a \le 2$ by (24). □

8.2. Switching and variance.

Let $\mathcal{C}_m$ be the P_3-decompositions of K_m. For uniform $P \in \mathcal{C}_m$, write $C_i(P)$ for the number of paths centered at i. Then

$$\sum_i C_i(P) = \frac{m(m-1)}{4}, \qquad \mathbb{E}C_i = \frac{m-1}{4}. \tag{62}$$

The number $\mathcal{B}_m$ of partial decompositions leaving two disjoint edges exposed is

$$\mathcal{B}_m = 3\binom{m}{4} b = \frac{m(m-1)(m-2)(m-3)}{8}\, b. \tag{63}$$

Lemma 8.2. *One has*

$$4\mathcal{B}_m = \sum_{P \in \mathcal{C}_m} \left[\sum_i 2C_i(P)(m-1-2C_i(P)) - \frac{m(m-1)}{4} \right]. \tag{64}$$

Proof. Choose an edge ia in a path $\{ia, ic\}$ centered at i and an edge ib in a path $\{ib, bd\}$ centered at b. There are $2C_i(m-1-2C_i)$ choices. Replace these paths by $\{ic, ib\}$, exposing ia, bd. The exposed edges are disjoint unless $d = a$.

Each path $\{bi, ba\}$ of P supplies exactly one exceptional choice: the edge ia is centered at either i or a, determining the choice based at that endpoint. There are therefore $m(m-1)/4$ exceptions.

Conversely, each of the four cross edges between the two disjoint exposed edges lies in a path of the partial decomposition. Its center determines one reversal. Every decomposition counted by $\mathcal{B}_m$ therefore has four preimages. □

Theorem 8.3. *For a uniform P_3-decomposition of K_m,*

$$\frac{b}{a} = \frac{(m-1)(m-2) - 16\,\mathrm{Var}(C_1)}{(m-1)(m-3)}. \tag{65}$$

The conditions $n_+(W(T(m))) = 1$, $(m-3)b \geq (m-4)a$, and $\mathrm{Var}(C_1) \leq (m-1)/8$ are equivalent. They are also equivalent to $\rho \geq m^2(m-1)/16$, or to $\alpha \leq 4/m$.

Proof. By (62),

$$\mathbb{E}\left[\sum_i 2C_i(m-1-2C_i) - \frac{m(m-1)}{4}\right] = m\left(\frac{(m-1)(m-2)}{4} - 4\,\mathrm{Var}(C_1)\right).$$

Divide (64) by $M = 2(m-2)a$ and use (63). The equivalences follow from (60), Theorem 8.1, and (24). □

8.3. **An injection between decompositions.**

Theorem 8.4. *For every admissible m, the probability in* (57) *satisfies $\eta_H \leq 1/2$. Consequently,* $\mathrm{Var}(C_1) \leq (m-1)/8$.

Proof. For $m = 4$, pairing two edges at a vertex leaves only one other incident edge, which cannot also be centered there. Assume $m \geq 5$.

Fix $v \neq c$. Consider pairs (P, A), where $P \in \mathcal{C}_m$ and A is a marked path of P centered at v and avoiding vc. We inject the pairs with vc centered at v into those with vc centered at c.

Starting with a pair in the domain, define matchings on the vertex labels of K_m by

$$xy \in \mathcal{M}_v \iff \{vx, vy\} \text{ is a path of } P \text{ centered at } v,$$
$$xy \in \mathcal{M}_c \iff \{cx, cy\} \text{ is a path of } P \text{ centered at } c.$$

The label c has degree one in $\mathcal{M}_v$ and zero in $\mathcal{M}_c$. Its component in their union is an alternating path $x_0 = c, x_1, \ldots, x_L = z$, starting in $\mathcal{M}_v$. It cannot contain v, since vc is already centered at v.

The corresponding paths of P are

$$\{vx_{i-1}, vx_i\} \quad (i \text{ odd}), \qquad \{cx_{i-1}, cx_i\} \quad (i \text{ even}). \tag{66}$$

At z, the edge not used in (66) is cz for odd L and vz for even L. It must be centered at z. Its path is therefore $\{cz, zy\}$ or $\{vz, zy\}$ for some y.

Replace (66) and this last path by

$$\{vc, cx_1\}, \qquad \begin{cases} \{vx_{i-1}, vx_i\}, & 2 \leq i \leq L,\ i \text{ even}, \\ \{cx_{i-1}, cx_i\}, & 3 \leq i \leq L,\ i \text{ odd}, \end{cases} \tag{67}$$

followed by $\{vz, zy\}$ for odd L or $\{cz, zy\}$ for even L. Every affected base edge still occurs exactly once, and vc is now centered at c.

For the marker, write

$$\Gamma_j = \{vx_{2j}, vx_{2j+1}\}, \qquad \Gamma'_j = \{vx_{2j+1}, vx_{2j+2}\}$$

where defined. Since Γ_0 contains vc, it is not marked. If $A = \Gamma_j$ for $j \geq 1$, mark Γ'_{j-1}; otherwise keep A.

In the image, v has degree one in $\mathcal{M}_c$ and zero in $\mathcal{M}_v$. Its alternating component recovers the shifted path and the path at z. Reversing the shift and the marker rule recovers (P, A), proving injectivity.

The stabilizer of v, c is transitive on the possible markers. The inequality between total counts therefore holds for each prescribed marker, giving $\eta_H \leq 1/2$. Apply Corollary 7.2 with $d = m - 1$. □

Corollary 8.5 (Triangular graphs)**.** *For every $m \geq 4$ with $m \equiv 0, 1 \pmod 4$,*

$$\frac{m-4}{m-3} < \frac{b}{a} < 2, \qquad \mathrm{In}(W(T(m))) = (1, \tbinom{m}{2} - 1, 0).$$

Proof. For the lower bound, we show that the preceding injection is not surjective. At $m = 4$, $\eta_H = 0$. For $m \geq 5$, choose distinct v, c, x, z, y and prescribe

$$\{vc, cx\}, \qquad \{vx, vz\}, \qquad \{cz, zy\},$$

marking the middle path. The remaining graph has even size and connected nonisolated part. For $m = 5$, its edges vy, cy, xy, xz form a tree; for $m > 5$, each additional vertex is adjacent to all five labels. Sumner's theorem [12] gives a decomposition of the remainder.

Here vc is centered at c. The alternating component is v, x, z, with its last path centered at v marked. Reversing the shift leaves only one v-centered path on the original alternating path, and that path contains vc, so cannot be marked. A marker elsewhere would stay there. The prescribed pair is therefore outside the image. Transitivity gives $\eta_H < 1/2$, and (65) gives the strict lower bound.

For the upper bound, recall that incident exposed edges gave $c_3 \geq 1$. At $m = 4$, there are two common neighbors and $\mathcal{R}$ is empty, so $c_3 \geq 2$. For $m \geq 5$, expose $12, 13$ and prescribe $\{14, 15\}$. Removing these four base edges leaves even size and a connected nonisolated part, which again has a decomposition. The prescribed path contributes two to c_3; the common neighbor 23 contributes at least one more because it cannot be matched into $\mathcal{R}$. Thus some state has $c_3 \geq 3$, making the average inequality in Theorem 8.1 strict. Hence $b < 2a$, and (22) gives the inertia. □

9. Strongly regular graphs with least eigenvalue -2

Seidel's list consists of the cocktail-party, lattice, and triangular families, and the Petersen, Shrikhande, three Chang, Schläfli, and 10-regular Clebsch graphs [11, 2, 3]. The Schläfli graph has odd order. For the other exceptions, the following bound avoids any assumption of rank-three symmetry.

Lemma 9.1. *Let A be the adjacency matrix of a strongly regular graph with parameters n, k, r, s. For symmetric W and real a_0, b_0, put $W_0 = a_0 A + b_0(J - I - A)$ and $E = W - W_0$. If*

$$\max\{ra_0 - (r+1)b_0,\ sa_0 - (s+1)b_0\} + \max_u \sum_v |E_{uv}| < 0, \tag{68}$$

then W is negative definite on $\mathbf{1}^\perp$. If also $W \neq 0$ and W is entrywise nonnegative, then $\mathrm{In}(W) = (1, n-1, 0)$.

Proof. The first maximum bounds W_0 on $\mathbf{1}^\perp$, and symmetry gives $\|E\|_2 \leq \max_u \sum_v |E_{uv}|$. This proves negativity on a subspace of dimension $n - 1$. The remaining direction is positive because $\mathbf{1}^\mathsf{T} W \mathbf{1} > 0$. □

Proposition 9.2. *The even-order exceptional graphs have cofactor inertia* $(1, n-1, 0)$ *and the following perfect matching counts.*

Graph	n	$\mathrm{pm}(G)$
Petersen	*10*	*6*
10-*regular Clebsch*	*16*	*89 760*
Shrikhande	*16*	*1 920*
Chang 1	*28*	*3 322 626 048*
Chang 2	*28*	*3 322 632 960*
Chang 3	*28*	*3 322 598 400*

Proof. Appendix A gives the graph models and integer recurrence used for all counts below. The supplied certificates contain the full matrices, which the verifier recomputes.

For Petersen, $W = 2(J - I)$. For the 10-regular Clebsch graph, $a = 8976$, $b = 8208$, and $r = 2, s = -2$; both nonprincipal cofactor eigenvalues are negative.

For Shrikhande, adjacent cofactors are 320 and nonadjacent cofactors belong to $\{272, 280\}$. Take $W_0 = 320A + 276(J - I - A)$. Since $r = 2, s = -2$, its largest eigenvalue on $\mathbf{1}^\perp$ is -188. Each error row has at most nine entries of absolute value four, so

$$x^{\mathsf{T}} W x \le (-188 + 36)\|x\|^2 = -152\|x\|^2 \qquad (x \perp \mathbf{1}).$$

All three Chang graphs have $(n, k, r, s) = (28, 12, 4, -2)$, and their exact cofactors satisfy

$$276\,872\,832 \le W_{uv} \le 276\,925\,824 \qquad (uv \in E),$$
$$259\,084\,224 \le W_{uv} \le 259\,130\,880 \qquad (u \ne v,\ uv \notin E).$$

Use $W_0 = 276\,900\,000A + 259\,100\,000(J - I - A)$. Its largest eigenvalue on $\mathbf{1}^\perp$ is $4(276\,900\,000) - 5(259\,100\,000) = -187\,900\,000$. The error bound is

$$\|W - W_0\|_2 \le 12(27\,168) + 15(30\,880) = 789\,216,$$

so $x^{\mathsf{T}} W x \le -187\,110\,784\|x\|^2$ for $x \perp \mathbf{1}$. Apply Lemma 9.1. □

Proof of Theorem 1.2. For $K_{t\times 2}$ with $t \ge 3$, Theorem 5.2 gives $a \ge b > 0$: deleting one whole part leaves a perfect matching. Its nonprincipal cofactor eigenvalues $-b$ and $b - 2a$ are strictly negative. At $t = 2$, the graph is C_4 with inertia $(1, 1, 2)$.

Even-order lattice graphs have even m; $R_2 = C_4$, and Theorem 6.8 treats $m \ge 4$. Even-order triangular graphs have $m \ge 4$ with $m \equiv 0, 1 \pmod 4$, so Corollary 8.5 applies. Perfect matchings were established in each family. Proposition 9.2 covers every even-order exception in Seidel's list. □

10. Limits and open questions

The degree bound in Theorem 1.1 is necessary even when all bridges are pendant.

Proposition 10.1. *A connected graph of maximum degree four can have only pendant bridges while its line graph has two positive cofactor eigenvalues.*

Proof. Take $E(H) = \{02, 03, 04, 05, 12, 13\}$. Its only bridges, $04, 05$, are pendant. In the order $04, 05, 03, 13, 12, 02$, exact deletion gives

$$W(L(H)) = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 3 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$(\lambda+2)(\lambda^2+4\lambda+2)(\lambda^3-6\lambda^2+2\lambda+4).$$

The cubic has roots in $(-1,0),(1,2),(5,6)$; the other roots are negative. Thus the inertia is $(2,4,0)$. The line graph has least adjacency eigenvalue -2, since it contains an induced C_4 and $A+2I \succeq 0$. □

Nor does a Lorentzian polynomial $h_{2,G}$ imply that all $h_{j,G}$ are Lorentzian: Theorem 3.10 gives a failure at $j=4$, even for a three-edge-connected cubic base. Theorem 3.9 also rules out bounding the positive cofactor index from the least adjacency eigenvalue alone.

For strongly regular graphs, the case below least eigenvalue -2 remains open.

Conjecture 10.2. *Let G be a primitive rank-three strongly regular graph of even order with a perfect matching. If $W = aA + b(J-I-A)$ and $r > s$ are the nonprincipal adjacency eigenvalues, then*

$$\frac{r}{r+1} \le \frac{b}{a} \le \frac{s}{s+1}.$$

The 5-regular Clebsch graph satisfies this: $a=141$, $b=90$, and its adjacency eigenvalues are $5,1,-3$, giving $b/a = 30/47 \in (1/2, 3/2)$.

Conjecture 10.3. *Let H be connected and d-regular, with $d>2$ and an even number of edges. For a uniform P_3-decomposition and an independent uniform vertex U,*

$$\operatorname{Var}(C_U) \le d/8, \qquad \textit{equivalently} \qquad \bar{\eta}_H \le 1/2.$$

The bound holds for the complete, balanced complete bipartite, and cubic bases treated above. The general regular case, cofactor inertia for higher-degree line-graph bases, and the strongly regular case beyond Theorem 1.2 are not settled here.

Appendix A. Exact computation

For an even vertex set S, choose its least-labeled vertex u and compute

$$f(\varnothing)=1, \qquad f(S) = \sum_{v \in N_G(u) \cap S} f(S \setminus \{u,v\}). \tag{69}$$

Exposing the partner of u proves $f(S) = \operatorname{pm}(G[S])$. Memoization on vertex bit masks gives $M = f(V)$ and $W_{uv} = f(V \setminus \{u,v\})$.

Petersen has the 2-subsets of $\{0,\dots,4\}$ as vertices, adjacent when disjoint. The Clebsch models use the even subsets, adjacent at symmetric difference two for the 10-regular graph and four for the 5-regular graph. Shrikhande is the Cayley graph on $\mathbb{Z}_4^2$ with connection set $\{\pm(1,0),\pm(0,1),\pm(1,1)\}$.

For Chang, start with $T(8)$ on lexicographically ordered 2-subsets of $\{0,\dots,7\}$. Complement the adjacencies between S and its complement, using

$$\begin{aligned} S_1 &= \{01,23,45,67\},\\ S_2 &= \{01,12,02,34,45,56,67,37\},\\ S_3 &= \{01,12,23,34,45,56,67,07\}. \end{aligned}$$

These are the standard $4K_2$, C_3+C_5, and C_8 switching models [4]. The verifier checks their strongly regular parameters, recomputes all cofactors, and checks the bounds in Proposition 9.2.

The C++ recurrence accepts at most 28 vertices and checks for overflow in unsigned 64-bit arithmetic. Even K_{28} has only $27!! = 213\,458\,046\,676\,875 < 2^{64}$ perfect matchings. This implementation limit does not restrict the structural formulas.

The package checks those formulas against independent matching recursion on all 55 eligible connected subcubic graphs in the NetworkX atlas (order at most seven), and reconstructs both the

unbounded family and the example with four monomers. Finite inertias are checked by exact rational symmetric elimination, without numerical eigensolvers. Seeded random cases provide further regression tests. These finite checks supplement the arbitrary-order proofs.

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