Clique minors and fractional clique covers in graphs with independence number two

Mantis Discovery

Abstract

For graphs with independence number at most two, we convert bounds on clique minors in terms of order into chromatic bounds without accumulating an error across the join factors of a critical graph. Every such graph G has an odd-order induced subgraph F and an integer $b \geq 0$ such that $\chi(G) = b + (|F|+1)/2$ and $h(G) \geq b + h(F)$. Consequently, the largest hereditary deficit $\chi - h$ equals the largest hereditary deficit $\lceil |V|/2 \rceil - h$. Combining this reduction with the Chudnovsky–Seymour half-order bound gives

$$h(G) \geq \left\lfloor \frac{2 + 4\min\{1/4, 1/T\}}{3} \left(\chi(G) - \frac{1}{2} \right) \right\rfloor \qquad \text{when } \theta_f(G) \leq T.$$

Thus fractional clique covering number at most four implies $h(G) \geq \chi(G) - 1$, while $\theta_f(G) < 31/8$ implies Hadwiger's inequality. For each $T < 4$, any counterexample with $\theta_f(G) \leq T$ contains an induced counterexample whose order is bounded in terms of T. Applied to the complement H of the Higman–Sims graph, the method yields $h(H[w]) \geq \lfloor (24\chi(H[w]) - 12)/25 \rfloor$ for every clique inflation and $h(H[S]) \geq \chi(H[S]) - 2$ for every induced subgraph.

1 Introduction

For a finite simple graph G, let $h(G)$ be its Hadwiger number: the largest order of a complete graph that is a minor of G. A model of a complete minor consists of pairwise disjoint connected vertex sets, called branch sets, with an edge between every two sets. We write $\chi(G)$, $\omega(G)$ and $\alpha(G)$ for the chromatic, clique and independence numbers.

When $\alpha(G) \leq 2$, every color class has at most two vertices, so

$$\chi(G) \geq \left\lceil \frac{|V(G)|}{2} \right\rceil. \tag{1}$$

Thus the order bound $h(G) \geq \lceil |V(G)|/2 \rceil$ is weaker than Hadwiger's inequality for a particular graph. The two statements are equivalent when asserted throughout the class of graphs with independence number at most two; this is part of the classical reduction developed in [8]. For quantitative bounds, however, a direct sum over the join factors of a critical graph can add the same error many times. Our first result avoids that loss.

Theorem 1.1. *For every nonempty graph G with $\alpha(G) \leq 2$, there are an odd-order induced subgraph F of G and an integer $b \geq 0$ such that*

$$\chi(G) = b + \frac{|V(F)| + 1}{2}, \qquad h(G) \geq b + h(F). \tag{2}$$

The reduction transfers the chromatic deficit to one induced subgraph without adding a separate loss for each join factor. It uses Gallai's theorem on critical graphs [6], followed by an explicit construction between two join factors.

We apply Theorem 1.1 to fractional clique covers. The fractional clique covering number $\theta_f(G)$ is the minimum of $\sum_C \lambda_C$ over nonnegative weights on the cliques of G such that $\sum_{C \ni v} \lambda_C \geq 1$ for each vertex v. Equivalently, $\theta_f(G) = \chi_f(\overline{G})$. Define

$$a(T) = \frac{2 + 4\min\{1/4, 1/T\}}{3} \qquad (T \geq 1). \tag{3}$$

Theorem 1.2. *Let G be nonempty, with $\alpha(G) \leq 2$ and $\theta_f(G) \leq T$. Then*

$$h(G) \geq \left\lfloor a(T) \left(\chi(G) - \frac{1}{2} \right) \right\rfloor. \tag{4}$$

If $T \leq 4$, then $h(G) \geq \chi(G) - 1$. If $T < 31/8$, then $h(G) \geq \chi(G)$.

The quantitative estimate uses a half-order clique-minor bound of Chudnovsky and Seymour [4, Theorem 1.3]. The conclusion for $T < 31/8$ also uses Carter's computer-assisted theorem that every K_8-free graph with independence number at most two satisfies Hadwiger's inequality [3, Theorem 3]. Costa, Luu, Wood and Yip [5, Theorem 2.28] prove Hadwiger's inequality for all inflations when $\theta_f \leq 124/33$. Retaining the integer clique bound at odd orders raises the strict threshold to $31/8$; the argument leaves order 31 unresolved at the endpoint.

For a nonnegative integer vector $w = (w_v)_{v \in V(X)}$, the *clique inflation* $X[w]$ replaces v by a clique of size w_v, with complete adjacency between two replacement cliques exactly when their base vertices are adjacent. A zero weight deletes the vertex. Fractional clique covers pass to these inflations [5, Corollary 2.21]. Therefore Theorem 1.2 applies to an unbounded family of graphs for each fixed host, without a restriction on the ratios of its positive weights.

Our explicit application is the complement H of the Higman–Sims graph. A fractional clique cover of total weight $50/11$ and a complete-minor model of order 56 give the following conclusions.

Theorem 1.3. *For the Higman–Sims complement H,*

$$h(H[w]) \geq \left\lfloor \frac{24\chi(H[w]) - 12}{25} \right\rfloor \qquad \text{for every nonzero } w \in \mathbb{Z}_{\geq 0}^{100}, \tag{5}$$

$$h(H[S]) \geq \chi(H[S]) - 2 \qquad \text{for every } S \subseteq V(H). \tag{6}$$

Costa et al. construct a K_{50} model in H [5, Theorem 3.18]. Our computer-verified certificate has twelve singleton and forty-four edge branch sets, all listed in the appendix. Norin and Seymour [7] obtain dense minors rather than complete minors. The exact Hadwiger inequality for every inflation of H remains open.

2 Passing from order to chromatic number

The join $A \vee B$ of two graphs on disjoint vertex sets adds all edges between the sets. A nonempty graph is *vertex-critical* if deleting any vertex lowers its chromatic number. It is *indecomposable* if it is not a join of two nonempty graphs, or equivalently if its complement is connected.

We use Gallai's theorem: a vertex-critical graph of order less than $2\chi - 1$ is decomposable [6]. In the present class it has the following immediate consequence.

Lemma 2.1. *If Q is vertex-critical and indecomposable, with $\alpha(Q) \leq 2$, then $|V(Q)| = 2\chi(Q) - 1$.*

Proof. Put $k = \chi(Q)$ and $n = |V(Q)|$. A $(k-1)$-coloring of $Q - v$ has classes of size at most two, so $n - 1 \leq 2(k-1)$. Thus $n \leq 2k - 1$. Gallai's theorem gives the reverse inequality. □

The next construction does not require either factor to be critical.

Lemma 2.2. *Let A and B have odd orders $a \geq b$. In $A \vee B$ there are b pairwise adjacent branch sets, disjoint from an induced subgraph A' of A of order $a - b + 1$, such that every branch set is adjacent to every vertex of A'. Each branch set has size at most two.*

Proof. Choose a vertex $v \in V(B)$. Pair the remaining $b-1$ vertices of B with distinct vertices of A, using the join edges. Take these $b - 1$ pairs and the singleton $\{v\}$ as branch sets. Two pairs are adjacent across the join. The singleton is adjacent to the A-endpoint of every pair. Every branch set contains a vertex of B, so it is adjacent to every unused vertex of A. Those unused vertices induce A'. □

Proof of Theorem 1.1. Choose a vertex-critical induced subgraph $Q \subseteq G$ with $\chi(Q) = \chi(G)$. The connected components of $\overline{Q}$ give a decomposition $Q = Q_1 \vee \cdots \vee Q_k$ into indecomposable factors. Each factor is vertex-critical, because the chromatic number of a join is the sum of the chromatic numbers of its factors. By Lemma 2.1, every factor has odd order n_i and

$$\chi(G) = \sum_{i=1}^{k} \frac{n_i + 1}{2}.$$

If $k = 1$, take $F = Q$ and $b = 0$.

Otherwise apply Lemma 2.2 to two factors, of orders $a \geq b_0$. Remove the b_0 constructed branch sets and retain the odd-order residual factor of order $a - b_0 + 1$. The branch sets are adjacent to every residual vertex, including those in all the other join factors. The numerical identity

$$\frac{a+1}{2} + \frac{b_0 + 1}{2} = b_0 + \frac{a - b_0 + 2}{2}$$

preserves the sum after recording the b_0 extracted branch sets. Repeat until one odd-order induced subgraph F remains. No criticality assertion is needed for a residual factor: only its odd order enters subsequent steps.

All extracted branch sets are pairwise adjacent. A set extracted earlier is adjacent to every vertex remaining at that stage, and hence to all later branch sets and to every vertex of F. If their total number is b, the identities give $\chi(G) = b + (|V(F)| + 1)/2$. Appending any complete-minor model in F gives $h(G) \geq b + h(F)$. □

For a graph X, write $[x]_+ = \max\{x, 0\}$ and define

$$D_\chi(X) = \max_{\varnothing \neq S \subseteq V(X)} [\chi(X[S]) - h(X[S])]_+,$$
$$D_{1/2}(X) = \max_{\varnothing \neq S \subseteq V(X)} [\lceil |S|/2 \rceil - h(X[S])]_+.$$

Both quantities are zero for the empty graph.

Corollary 2.3. *If $\alpha(X) \leq 2$, then $D_\chi(X) = D_{1/2}(X)$. The maximum defining $D_{1/2}$ can be restricted to odd sets S.*

Proof. Inequality (1) gives $D_{1/2} \le D_\chi$. For each nonempty induced subgraph G of X, Theorem 1.1 gives an odd induced $F \subseteq G$ with

$$\chi(G) - h(G) \le \frac{|V(F)|+1}{2} - h(F).$$

Take positive parts and then the maximum over G. □

We will also use the multiplicative version.

Corollary 2.4. *Suppose $\alpha(X) \le 2$, $0 \le a \le 1$, and $c \in \mathbb{R}$. If every nonempty odd-order induced subgraph F of X satisfies*

$$h(F) \ge \left\lfloor a\frac{|V(F)|+1}{2} + c \right\rfloor,$$

then every nonempty induced subgraph G of X satisfies $h(G) \ge \lfloor a\chi(G) + c \rfloor$.

Proof. Use (2) and put $q = (|V(F)|+1)/2$. Since b is an integer,

$$h(G) \ge b + \lfloor aq + c \rfloor = \lfloor a\chi(G) + c + (1-a)b \rfloor \ge \lfloor a\chi(G) + c \rfloor.$$

□

3 Fractional clique covers

The graph-theoretic input is the following result of Chudnovsky and Seymour [4, Theorem 1.3]:

$$\begin{aligned} \alpha(G) \le 2, \qquad \omega(G) &\ge \begin{cases} |V(G)|/4, & |V(G)| \text{ even}, \\ (|V(G)|+3)/4, & |V(G)| \text{ odd} \end{cases} \\ \implies \quad h(G) &\ge \lceil |V(G)|/2 \rceil. \end{aligned} \tag{7}$$

Adding universal vertices gives the following interpolated consequence.

Lemma 3.1. *For every nonempty G with $\alpha(G) \le 2$, writing $n = |V(G)|$ and $\omega = \omega(G)$,*

$$h(G) \ge \min\left\{ \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n+2\omega}{3} \right\rfloor \right\}. \tag{8}$$

Proof. Let q be the right side. First suppose $q \le \lfloor n/2 \rfloor$ and set $t = n - 2q$. The graph $G \vee K_t$ has even order $n+t$ and clique number $\omega + t$. Moreover

$$4(\omega + t) - (n + t) = 4\omega + 2n - 6q \ge 0.$$

By (7) it has a complete minor of order $(n+t)/2 = n - q$. Delete the t added vertices from this model, discarding each branch set that meets an added vertex. At most t branch sets are discarded. The remaining model lies in G and has at least $(n-q) - t = q$ branch sets.

The other case has odd n and $q = (n+1)/2$. The inequality $q \le (n+2\omega)/3$ gives $4\omega \ge n+3$, so (7) applies directly to G. □

If $\theta_f(G) \le T$, then each induced subgraph F satisfies

$$\omega(F) \ge \frac{|V(F)|}{T}. \tag{9}$$

This is the standard fractional clique-cover inequality [5, Observation 2.18(3)]. The corresponding monotonicity under clique inflation is [5, Corollary 2.21].

Proof of (4). Put $r = \min\{1/4, 1/T\}$, so $a(T) = (2+4r)/3 \le 1$. If F has odd order $2q-1$, Lemma 3.1 and (9) give

$$h(F) \ge \min\left\{q, \left\lfloor \frac{(1+2r)(2q-1)}{3} \right\rfloor\right\} = \left\lfloor a(T)\left(q - \frac{1}{2}\right)\right\rfloor.$$

The equality follows from $a(T) \le 1$. Apply Corollary 2.4 with $c = -a(T)/2$. For $T \le 4$, the coefficient $a(T)$ is one, and the bound is $\chi(G) - 1$. □

The same argument bounds the order of a possible counterexample below the threshold four.

Theorem 3.2 (Bound on the order of a counterexample)**.** *Let $\alpha(G) \le 2$ and $\theta_f(G) \le T < 4$. If $h(G) < \chi(G)$, then G contains an induced subgraph F such that*

$$|V(F)| \equiv 3 \pmod 4, \qquad 31 \le |V(F)| \le \frac{T}{4-T}, \qquad h(F) < \frac{|V(F)|+1}{2}. \tag{10}$$

In particular, $\theta_f(G) < 31/8$ implies $h(G) \ge \chi(G)$.

Proof. By Theorem 1.1, a failure of $h(G) \ge \chi(G)$ supplies an odd induced subgraph F with $h(F) < (|V(F)|+1)/2$. Put $n = |V(F)|$. Carter's theorem [3] implies $\omega(F) \ge 8$: otherwise F would satisfy Hadwiger's inequality, and hence the half-order bound. If $n \le 29$, then $8 \ge (n+3)/4$, contradicting (7). Thus $n \ge 31$.

Since $T < 4$, inequality (9) gives $\omega(F) > n/4$. If $n \equiv 1 \pmod 4$, integrality gives $\omega(F) \ge (n+3)/4$, again contradicting (7). Therefore $n \equiv 3 \pmod 4$. To fail (7) in this residue class, the integer $\omega(F)$ must satisfy

$$\frac{n}{T} \le \omega(F) \le \frac{n+1}{4}.$$

Rearranging gives $n \le T/(4-T)$ and proves (10). When $T < 31/8$, the upper endpoint is less than 31, so the interval is empty. □

This proves the last assertion of Theorem 1.2. For each fixed $T < 4$, any counterexample contains an induced counterexample of one of the finitely many odd orders in (10). For example, $T = 39/10$ leaves only the orders $31, 35, 39$.

Corollary 3.3. *Let X be a graph with $\alpha(X) \le 2$ and $\theta_f(X) \le T$. Every nonempty clique inflation $X[w]$ satisfies* (4)*, with $G = X[w]$. If $\theta_f(X) < 31/8$, all such inflations satisfy Hadwiger's inequality.*

Proof. Inflation preserves independence number at most two, and [5, Corollary 2.21] gives $\theta_f(X[w]) \le \theta_f(X)$. Apply Theorem 1.2. □

4 Averaging over automorphisms

For an integer weight vector $w \ge 0$, put

$$W = \sum_v w_v, \qquad S_t = \{v : w_v \ge t\}, \qquad n_t = |S_t|.$$

For a graph X, write

$$\mathrm{TV}_X(w) = \sum_{uv \in E(X)} |w_u - w_v|.$$

This quantity records the variation of the weights across its edges.

Theorem 4.1. *Let X be vertex- and edge-transitive, with N vertices and $m > 0$ edges. Suppose X has a complete-minor model with s singleton and p edge branch sets. Then*

$$h(X[w]) \geq \left\lceil \frac{s}{N}W + \frac{p}{m} \sum_{uv \in E(X)} \min(w_u, w_v) \right\rceil \tag{11}$$

$$= \left\lceil \frac{s+p}{N}W - \frac{p}{2m}\,\mathrm{TV}_X(w) \right\rceil \qquad (w \in \mathbb{Z}_{\geq 0}^{V(X)}). \tag{12}$$

Proof. Fix an automorphic image of the model. A singleton $\{v\}$ lifts to w_v singleton clones. An edge uv lifts to $\min(w_u, w_v)$ disjoint pairs of clones, one over each endpoint. Branch sets lifted from the same edge are adjacent through its complete bipartite clone graph. Singleton clones over one vertex are adjacent inside its replacement clique. Sets lifted from different base branch sets remain adjacent through an edge joining those base sets. Thus they form a complete-minor model.

For a uniformly chosen automorphism, each singleton is uniform on $V(X)$ and each edge is uniform on $E(X)$. The expectation of the lifted order is the expression in (11). Some image attains at least this mean, and its order is an integer.

The graph X is d-regular with $m = Nd/2$. The identity

$$\sum_{uv \in E(X)} \min(w_u, w_v) = \frac{dW - \mathrm{TV}_X(w)}{2}$$

gives (12). □

The expectation selects one automorphic image of the integral model; no branch sets from different images are combined.

Corollary 4.2. *Under the hypotheses of Theorem 4.1, let λ be the least adjacency eigenvalue of X. Then*

$$h(X[w]) \geq \left\lceil \frac{p(d-\lambda)}{2mN} \sum_{t \geq 1} n_t^2 + \left(\frac{s}{N} + \frac{p\lambda}{2m} \right) W \right\rceil. \tag{13}$$

Proof. For $S \subseteq V(X)$ with $|S| = n$, write $\mathbf{1}_S = (n/N)\mathbf{1} + x$, where $x \perp \mathbf{1}$. The Rayleigh bound gives

$$2e(X[S]) \geq \frac{d}{N}n^2 + \lambda\left(n - \frac{n^2}{N}\right) = \frac{d-\lambda}{N}n^2 + \lambda n.$$

This is the usual one-set spectral bound; see [2]. Now use

$$\sum_{uv \in E(X)} \min(w_u, w_v) = \sum_{t \geq 1} e(X[S_t]), \qquad \sum_{t \geq 1} n_t = W,$$

and substitute into (11). □

Averaging a perfect matching in the complement supplies a coloring bound.

Proposition 4.3. *Suppose $\overline{X}$ is k-regular and edge-transitive, with a perfect matching. Then*

$$\chi(X[w]) \leq \left\lfloor \frac{W}{2} + \frac{\mathrm{TV}_{\overline{X}}(w)}{2k} \right\rfloor. \tag{14}$$

Proof. For any perfect matching M of $\overline{X}$, the two replacement cliques over an edge $uv \in M$ are anticomplete in $X[w]$. They can share a palette of $\max(w_u, w_v)$ colors. Use disjoint palettes for different matching edges. The resulting coloring uses

$$\sum_{uv \in M} \max(w_u, w_v) = \frac{W}{2} + \frac{1}{2} \sum_{uv \in M} |w_u - w_v|$$

colors. Under a uniform automorphism of $\overline{X}$, each edge of a fixed perfect matching is uniformly distributed over its $Nk/2$ edges. Thus the mean of the last sum is $\mathrm{TV}_{\overline{X}}(w)/k$. Some integral coloring bound is at most the floor of this mean. □

5 The Higman–Sims complement

Let B be the Higman–Sims graph and $H = \overline{B}$. We use the parameters and symmetry in Brouwer's account [1]:

$$B = \mathrm{srg}(100, 22, 0, 6), \qquad H = \mathrm{srg}(100, 77, 60, 56).$$

The automorphism group has rank three, hence both graphs are vertex- and edge-transitive. Their spectra are

$$B: \quad 22^1,\ 2^{77},\ (-8)^{22}; \qquad H: \quad 77^1,\ 7^{22},\ (-3)^{77}.$$

The graph B is triangle-free, so $\alpha(H[w]) \le 2$ for every inflation. Two estimates will be used separately: a fractional clique cover controls all inflations, while an explicit complete-minor model gives the stronger hereditary bound for induced subgraphs.

Every neighborhood $N_B(v)$ is a clique of H. Each vertex belongs to exactly 22 of these one hundred cliques. Giving each neighborhood weight $1/22$ yields

$$\theta_f(H) \le \frac{50}{11}, \qquad \omega(H[S]) \ge \left\lceil \frac{11|S|}{50} \right\rceil. \tag{15}$$

The fractional bound (5) now follows from Corollary 3.3, because $a(50/11) = 24/25$.

Proposition 5.1 (A computer-verified minor)**.** *The graph H has a K_{56} model with twelve singleton and forty-four edge branch sets. The graph B has a perfect matching.*

Proof. Section A gives a deterministic graph labeling, the branch sets and a perfect matching. The branch sets partition all one hundred vertices. The supplied verifier reconstructs the labeled graph, checks connectivity, and tests all $\binom{56}{2} = 1540$ required branch adjacencies using integer arithmetic. The proposition depends on that finite computation; it does not use numerical approximation or an unlisted search. □

Corollary 4.2 with $N = 100$, $m = 3850$, $d = 77$, $\lambda = -3$, $s = 12$ and $p = 44$ gives

$$h(H[w]) \ge \left\lceil \frac{4}{875} \sum_{t \ge 1} n_t^2 + \frac{18}{175} W \right\rceil. \tag{16}$$

In particular, for an induced subgraph of order n,

$$h(H[S]) \ge A(n) := \left\lceil \frac{4n^2 + 90n}{875} \right\rceil. \tag{17}$$

On the other hand, Lemma 3.1 and (15) give, for odd n,

$$h(H[S]) \ge C(n) := \min\left\{ \frac{n+1}{2}, \left\lfloor \frac{n + 2\lceil 11n/50 \rceil}{3} \right\rfloor \right\}. \tag{18}$$

Proof of (6). Let $L(n) = \max\{A(n), C(n)\}$. The exact evaluations in Table 1 show

$$\max_{1 \le n \le 99,\ n \text{ odd}} \left[\frac{n+1}{2} - L(n) \right]_+ = 2.$$

Thus every odd induced subgraph has a complete minor within two of the half-order target. Apply Corollary 2.3. The empty induced subgraph satisfies the conclusion directly. □

Loss bound	Odd orders n
0	$1, 5, 87, 89, 91, 93, 95, 97, 99$
1	$3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31,$ $33, 35, 37, 39, 41, 43, 47, 51, 55, 83, 85$
2	$45, 49, 53, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 77, 79, 81$

Table 1: The value $[(n+1)/2 - \max\{A(n), C(n)\}]_+$ at every odd order from 1 to 99. These are arithmetic bounds, not observed deficits of graphs.

5.1 Nonuniform weights

The averaging bounds give a condition that depends on the locations of the weights, in addition to (5).

Corollary 5.2. *For every weight vector $w \ge 0$,*

$$h(H[w]) \ge \left\lceil \frac{14W}{25} - \frac{\mathrm{TV}_H(w)}{175} \right\rceil, \tag{19}$$

$$\chi(H[w]) \le \left\lfloor \frac{W}{2} + \frac{\mathrm{TV}_B(w)}{44} \right\rfloor. \tag{20}$$

Consequently, Hadwiger's inequality holds whenever

$$\frac{\mathrm{TV}_H(w)}{175} + \frac{\mathrm{TV}_B(w)}{44} \le \frac{3W}{50}. \tag{21}$$

Proof. Use Theorem 4.1 and Proposition 4.3 and Proposition 5.1. Condition (21) makes the expression in the lower-bound ceiling at least the expression in the upper-bound floor. □

This criterion depends on where the weights differ, rather than only on their minimum and maximum. Two further families follow directly from the certificate.

Corollary 5.3. *Hadwiger's inequality holds for $H[w]$ in each of the following cases:*

1. *All weights are positive and $25 \max_v w_v \le 28 \min_v w_v$.*
2. *$w = t\mathbf{1} + z$, where $t \ge 1$ is an integer, $z \ge 0$ is integral, and z is supported on the twelve singleton vertices of an automorphic image of the model in Proposition 5.1.*

There is no upper bound on the nonzero entries of z in the second case.

Proof. Lifting the entire model uniformly gives $h(H[t\mathbf{1}]) \geq 56t$. A perfect matching of B colors $H[t\mathbf{1}]$ with $50t$ colors. Its independence number is at most two, so $\chi(H[t\mathbf{1}]) = 50t$.

For the first case, put $t = \min_v w_v$ and $M = \max_v w_v$. The uniform subgraph gives $h(H[w]) \geq 56t$, and the perfect matching colors $H[w]$ with at most $50M$ colors. The assumed ratio yields $50M \leq 56t$.

For the second case, lift all model branches t times and use every excess clone over a singleton as an additional singleton branch. All these sets remain pairwise adjacent, so $h(H[w]) \geq 56t + \sum_v z_v$. Color the uniform part with $50t$ colors and give each excess clone a fresh color. This gives $\chi(H[w]) \leq 50t + \sum_v z_v$. □

6 Further questions

The fractional-cover estimate separates two remaining questions. For $T < 4$, Theorem 3.2 bounds the orders of possible obstructions in terms of T alone. At $T = 4$, the argument gives an additive error of one but no finite order bound. Removing that last error would settle the fractional clique-cover threshold four without requiring a bound on the number of vertices.

For the Higman–Sims complement, the cover of weight $50/11$ applies to every inflation, whereas the explicit model is especially effective when many weights are comparable. The estimate (6) leaves at most two in the induced-subgraph problem. An improvement must either strengthen the half-order bound at the orders in Table 1, or use additional structure of the particular induced subgraph. The present certificate does not determine $h(H)$, and the averaging and cover bounds do not establish Hadwiger's inequality for every weight vector.

A An explicit model and its labeling

We specify the graph without depending on a graph library. This is the standard $1 + 22 + 77$ construction from the Steiner system $S(3, 6, 22)$ [1].

Work over $\mathbb{F}_2$ and let

$$g(x) = x^{11} + x^9 + x^7 + x^6 + x^5 + x + 1.$$

For every polynomial f of degree at most eleven, form $f(x)g(x)$ in positions $0, \ldots, 22$ and append an even-parity coordinate in position 23. Select the words of weight eight containing positions 22 and 23, and delete those two positions. The resulting seventy-seven six-subsets of $\{0, \ldots, 21\}$ form $S(3, 6, 22)$. Order them increasingly by the integer whose set bits are their members.

Label the graph vertices $0, 1, \ldots, 99$. Vertex 0 is the distinguished vertex, vertices $1, \ldots, 22$ are the points, and vertices $23, \ldots, 99$ are the ordered six-subsets. In B, join 0 to every point, a point to each block containing it, and two blocks when they are disjoint. There are no other edges. The verifier checks all 1540 triples in the Steiner property and every pair in the strongly regular parameter condition.

The twelve singleton branches in H are

$$1, 2, 3, 11, 12, 13, 14, 16, 18, 20, 21, 22.$$

The forty-four edge branches are

$$
\begin{array}{rrrr}
(4,86) & (5,76) & (6,63) & (7,58) \\
(8,35) & (9,91) & (10,95) & (17,71) \\
(19,83) & (23,66) & (24,62) & (25,75) \\
(26,55) & (28,65) & (27,30) & (31,47) \\
(32,40) & (33,77) & (34,87) & (36,68) \\
(37,82) & (38,56) & (41,52) & (43,99) \\
(44,78) & (46,98) & (0,48) & (45,50) \\
(51,85) & (15,53) & (59,97) & (29,60) \\
(61,74) & (69,92) & (72,94) & (73,88) \\
(64,79) & (57,80) & (39,81) & (67,84) \\
(49,89) & (42,90) & (70,93) & (54,96)
\end{array}
$$

A perfect matching of B in the same labeling is

$$
\begin{array}{rrrrr}
(0,5) & (1,98) & (2,49) & (3,95) & (4,94) \\
(6,99) & (7,92) & (8,91) & (9,88) & (10,93) \\
(11,89) & (12,86) & (13,85) & (14,81) & (15,90) \\
(16,83) & (17,87) & (18,74) & (19,78) & (20,77) \\
(21,76) & (22,84) & (23,70) & (24,72) & (25,68) \\
(26,82) & (27,75) & (28,67) & (29,62) & (30,65) \\
(31,80) & (32,71) & (33,79) & (34,63) & (35,69) \\
(36,66) & (37,73) & (38,61) & (39,64) & (40,60) \\
(41,54) & (42,58) & (43,59) & (44,56) & (45,52) \\
(46,57) & (47,55) & (48,53) & (50,97) & (51,96)
\end{array}
$$

The file `hs_certificate.json` records both lists. The script `verify_graphs.py` uses only the Python standard library. It reconstructs B and H, verifies all branch sets and the matching, and evaluates (17)–(18) at every odd order. It also checks the reduction on all 6228 labeled graphs with independence number at most two and at most six vertices. These finite tests do not prove the general reduction or the cited rank-three action.

References

[1] A. E. Brouwer, *Higman–Sims graph*, `https://aeb.win.tue.nl/graphs/Higman-Sims.html`.

[2] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, 2012. `10.1007/978-1-4614-1939-6`.

[3] D. Carter, *Hadwiger's conjecture with certain forbidden induced subgraphs*, Experimental Mathematics **35** (2026), no. 1, 92–110. `10.1080/10586458.2024.2428958`.

[4] M. Chudnovsky and P. Seymour, *Packing seagulls*, Combinatorica **32** (2012), no. 3, 251–282. `10.1007/s00493-012-2594-2`.

[5] J. Costa, E. Luu, D. R. Wood and J. H. Yip, *Verifying Hadwiger's conjecture for examples of graphs with* $\alpha(G) = 2$, arXiv:2512.17114v2. `https://arxiv.org/abs/2512.17114`.

[6] T. Gallai, *Kritische Graphen II*, Publ. Math. Inst. Hungar. Acad. Sci. **8** (1963), 373–395.

[7] S. Norin and P. Seymour, *Dense minors of graphs with independence number two*, J. Combin. Theory Ser. B **176** (2026), 101–110. `10.1016/j.jctb.2025.08.005`.

[8] M. D. Plummer, M. Stiebitz and B. Toft, *On a special case of Hadwiger's conjecture*, Discuss. Math. Graph Theory **23** (2003), no. 2, 333–363. 10.7151/dmgt.1206.