Boundary spectra and weak collisional relaxation in an electrostatic gyrokinetic model

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Abstract

Can collisionless transport or weak collisions make unresolved states decay uniformly in a linear electrostatic gyrokinetic model? We show that quasineutral coupling does not remove the obstruction created by slow particles when the quadratic energy operator is coercive. Infinite-dimensional subspaces have uniformly vanishing field response, and the undriven spectrum is the closed left half-plane for zero inflow and the imaginary axis for periodic boundaries. Under a stated nondegeneracy condition on coupling to the remaining velocity region, asymptotically passing approximate eigenvectors on the imaginary axis have vanishing field and concentrate near the passing boundary. For a Maxwellian velocity diffusion and a one-species linearized Coulomb model, differentiated wave packets give residuals bounded by $C(\nu^{1/3} + \nu^{1/5}|\omega|^{2/5})$ at bounded frequencies. If one species also has the required low-energy drift parity and an even gyroaverage, field feedback and any fixed number of bounded scalar observations cannot yield a positive decay rate uniform in ν with a fixed semigroup bound. These conclusions concern the displayed linear realizations and the full distribution norm.

Keywords. Gyrokinetics; transport operators; essential spectrum; weak collisions; field feedback.

1 Introduction

Velocity moments can predict a field or flux without resolving the distribution function. Hermite–Laguerre formulations connect fluid descriptions to the kinetic equation as velocity resolution increases, while closures supply information lost at truncation [4, 11]. Asymptotic closures can likewise succeed for particular observables in a prescribed regime [6]. Here we ask a different question: do the omitted kinetic states decay uniformly in the norm of the full distribution? An affirmative answer would justify eliminating them independently of the initial state.

The collisionless obstruction is classical. In absorbing domains, long residence times produce half-plane spectra and disk spectra for the evolution operators, with the realized sets determined by trajectory geometry and the boundary operator [7, 8]. Gyrokinetics adds a nontrivial field constraint. Quasineutrality is local along the field line but integrates over velocity, so its range is an infinite-dimensional space of spatial profiles. Eliminating the field changes the coefficient of the time derivative rather than producing a compact perturbation of transport.

We treat this coupling directly. On a bounded field-line interval, low-energy passing particles have speed of order $\sqrt{c}$. Choosing the characteristic phase to cancel the frequency and magnetic drift leaves a transport residual of order $\sqrt{c}$. The accessible velocity volume is of order $c^{3/2}$, and Cauchy–Schwarz then bounds the field response by $O(c^{3/4})$. The estimate holds on an infinite-dimensional subspace, which allows the construction to satisfy any fixed number of linear constraints.

A second construction, localized near outflow, approximates every spectral parameter with negative real part. When the drive vanishes, coercivity of the energy operator gives the reverse

inclusion: the absorbing generator has the closed left half-plane as its spectrum. For periodic boundaries, the same energy argument gives a unitary group with spectrum equal to the imaginary axis. Neither conclusion requires a magnetic well or an endpoint maximum of the field strength.

We also locate asymptotically passing approximate eigenvectors. If coupling to the complementary velocity region is nonzero at almost every field-line position, their fields converge weakly to zero. On any region with a fixed positive excess energy, the transport lifetime is bounded and the corresponding resolvent contribution is compact, which strengthens the field convergence and forces the sequence toward the boundary of the passing region. The argument allows the metric coefficient to vary along the field line.

Weak collisions introduce derivatives of the velocity-localized states. We estimate their cost for a Maxwellian Ornstein–Uhlenbeck model and for the gyrophase compression of a velocity-phase-conjugated, one-species linearized Landau operator. This yields a quantitative pseudospectral obstruction under any prescribed number of linear constraints. It does not identify collisional eigenvalues. Weak collisions can substantially change kinetic spectral type [13], and $\nu^{1/3}$ enhanced dissipation is known in other kinetic models [1, 2]; the present estimate retains both the field constraint and field-line geometry.

For feedback through the electrostatic field, a positive decay rate uniform in ν requires the observation or control gain to grow at least as $\nu^{-3/4}$ when the norm and semigroup bound remain fixed (Theorem 7.5). With a velocity-even gyroaverage, exact field cancellation rules out such a uniform rate for every gain, even after adding finitely many bounded scalar observations (Theorem 7.6). These are necessary conditions for the stated operators, not controller designs. Physical implementations of gyrokinetic Landau collisions introduce further modeling choices [10, 14].

All approximation lower bounds concern the full energy ball. They remain compatible with accurate fields or fluxes, smoother classes of initial data, and weaker topologies because the obstructing states have asymptotically negligible electrostatic field. The positive energy is consistent with the quadratic-form approach to gyrokinetic stability [5]. Our conclusions therefore concern linear full-state relaxation, not turbulent saturation or transport thresholds.

2 The operator and its boundary realizations

2.1 Phase space and the field constraint

Let $I = (a, b)$ be bounded. There are finitely many species, with mass $m_s > 0$ and charge $q_s \neq 0$. Let $B, \beta \in C^\infty(\overline{I})$ be positive, with positive lower bounds. The variables are field-line position $\theta \in I$, parallel velocity $v \in \mathbb{R}$, and magnetic moment $\mu > 0$. Set

$$E_s = \tfrac{1}{2} m_s v^2 + \mu B(\theta), \qquad F_s(E) = \left(\frac{m_s}{2\pi}\right)^{3/2} e^{-E}, \qquad \mathrm{d}^3 v_s = \frac{2\pi B(\theta)}{m_s}\,\mathrm{d}v\,\mathrm{d}\mu. \tag{2.1}$$

The temperature and density have been normalized to one. Write $\sigma = (B\beta)^{-1}$ and

$$\mathcal{H} = \bigoplus_s L^2\left(I \times \mathbb{R} \times (0,\infty), \frac{\sigma\,\mathrm{d}\theta\,\mathrm{d}^3 v_s}{F_s}\right), \qquad \mathcal{X} = L^2(I, \sigma\,\mathrm{d}\theta). \tag{2.2}$$

All inner products are conjugate-linear in the first argument. Norms without a subscript refer to the indicated space.

The real gyroaverage multiplier $J_s(\theta, v, \mu)$ is smooth locally and satisfies $|J_s| \le 1$. Define

$$(R\phi)_s = q_s F_s J_s \phi, \qquad R^* h = \sum_s q_s \int J_s h_s \, \mathrm{d}^3 v_s. \tag{2.3}$$

The adjoint is taken between the spaces in (2.2). Let C be bounded and self-adjoint on $\mathcal{X}$, with $C \ge c_- \mathrm{Id} > 0$. Quasineutrality and the quadratic energy operator are

$$\phi = Qh := C^{-1} R^* h, \qquad P = RQ = RC^{-1}R^*, \qquad M = \mathrm{Id} - P. \tag{2.4}$$

We assume that this energy is coercive:

$$m_0 \mathrm{Id} \le M \le \mathrm{Id} \qquad \text{for some } m_0 > 0. \tag{2.5}$$

It defines the equivalent Hilbert norm $\|h\|_M^2 = \langle h, Mh \rangle$.

Let Ω_s be a real drift frequency, smooth on $\overline{I} \times \mathbb{R} \times (0, \infty)$ locally in velocity. It may grow with energy. For the drive, assume real measurable d_s with

$$\sup_{\theta \in I} \sum_s \int |d_s|^2 F_s \, \mathrm{d}^3 v_s < \infty, \qquad (G\phi)_s = \mathrm{i} d_s F_s J_s \phi. \tag{2.6}$$

Thus $R, G : \mathcal{X} \to \mathcal{H}$ and $Q : \mathcal{H} \to \mathcal{X}$ are bounded. In particular, any drive polynomial in energy with uniformly bounded spatial coefficients is allowed.

The characteristic vector field and transport expression are

$$X_s = v\beta \partial_\theta - \frac{\mu}{m_s} \beta B' \partial_v, \qquad (T_0 h)_s = -X_s h_s - \mathrm{i}\Omega_s h_s. \tag{2.7}$$

The coupled evolution is

$$M \partial_t h = T_0 h + GQh, \qquad L = M^{-1}(T_0 + GQ), \qquad \mathcal{D}(L) = \mathcal{D}(T_0). \tag{2.8}$$

No compactness of R, Q, or P is assumed.

2.2 Absorption and periodicity

The flow of X_s preserves μ and E_s. It also preserves

$$\mathrm{d}m_s := \sigma \, \mathrm{d}\theta \, \mathrm{d}^3 v_s = \frac{2\pi}{m_s \beta} \, \mathrm{d}\theta \, \mathrm{d}v \, \mathrm{d}\mu. \tag{2.9}$$

Indeed, $X_s E_s = 0$ and the divergence of the weighted vector field in (θ, v) is zero. These identities also preserve $\mathrm{d}m_s / F_s$.

For absorption, let Φ_t be the characteristic flow and let $\tau_-(x)$ be its first backward exit time from the interval. The transport semigroup is

$$U_0(t) f(x) = \mathbf{1}_{\{t < \tau_-(x)\}} \exp\left(-\mathrm{i} \int_0^t \Omega(\Phi_{-r} x) \, \mathrm{d}r \right) f(\Phi_{-t} x). \tag{2.10}$$

Species indices are suppressed. This is a strongly continuous contraction semigroup. Its generator, denoted T_0, is the zero-inflow realization of (2.7). The incoming faces are $(a, v > 0)$ and $(b, v < 0)$. Formula (2.10) uses the first exit; a particle is not restored after a later re-entry. This characteristic definition avoids an unnecessary condition on the values of B at the endpoints. The general first-exit

construction is standard [8]; here measure preservation and the unit-modulus phase give contractivity directly.

For periodic boundaries, assume that B, β, J_s, Ω_s extend smoothly and periodically in θ, identify d_s periodically, and take the characteristic flow on the circle. Omitting the exit indicator in (2.10) gives a unitary group, whose generator T_0 is skew-adjoint. Periodicity is a hypothesis on the complete coefficient set. It is not asserted for a single nonzonal Fourier component whose coefficients change under a twist-and-shift connection [12].

For the local construction in Section 3, a third possibility is useful: any closed realization of (2.7) whose domain contains smooth compactly supported interior functions. That construction does not require this realization to generate a semigroup. The closure of the differential expression on such functions must not automatically be identified with the absorbing generator.

Proposition 2.1 (Generation and energy)**.** *For either the absorbing or periodic realization, L generates a strongly continuous semigroup $S(t)$ and*

$$\|S(t)\|_M \le e^{\gamma t}, \qquad \gamma = \|GQ\|_{\mathcal{H}\to\mathcal{H}}/m_0. \tag{2.11}$$

If $G = 0$, the absorbing semigroup is contractive in $\|\cdot\|_M$, whereas the periodic realization generates a unitary group in that norm.

Proof. First set $A = M^{-1}T_0$. It is closed and densely defined. Since $\Re\langle h, Ah\rangle_M = \Re\langle h, T_0 h\rangle \le 0$, it is dissipative. For $\lambda > 0$,

$$\lambda M - T_0 = [\mathrm{Id} - \lambda P(\lambda - T_0)^{-1}](\lambda - T_0), \qquad \|\lambda P(\lambda - T_0)^{-1}\| \le 1 - m_0 < 1.$$

Thus $\lambda - A$ is onto. The generation theorem for maximal dissipative operators gives a contraction semigroup in the energy norm [3]. The bounded operator $M^{-1}GQ$ has energy-norm bound at most $\|GQ\|/m_0$, yielding (2.11). For periodic boundaries, apply the same argument to $-T_0$. Both A and $-A$ are maximal dissipative, so A is skew-adjoint in the energy inner product and its group is unitary. □

Remark 2.2 (Two standard field constraints)*.* For equal-temperature singly charged ions and kinetic electrons, the nonzonal choice $C = 2\mathrm{Id}$ satisfies (2.5) if $\sup_\theta(\Gamma_i + \Gamma_e) < 2$, where $\Gamma_s = \int F_s J_s^2 \,\mathrm{d}^3 v_s$. One kinetic ion species with an adiabatic electron response has $M \ge \mathrm{Id}/2$. For the zonal modified-adiabatic response, $C = 2\mathrm{Id} - P_{\mathrm{av}}$, where P_{av} is the orthogonal projection onto constants in $\mathcal{X}$; then $\sup \Gamma_i < 1$ suffices. These statements follow from (2.3) and $\int F_s \,\mathrm{d}^3 v_s = 1$. Strict positivity is required at the chosen wavenumber, not uniformly as the wavenumber tends to zero.

3 Infinite-dimensional subspaces of quasimodes

3.1 A field estimate from velocity volume

For a measurable set A in phase space, let E_A denote multiplication by its indicator. Weighted Cauchy–Schwarz gives

$$\|QE_A\|^2 \le \|C^{-1}\|^2 \operatorname*{ess\,sup}_\theta \sum_s q_s^2 \int_{A_\theta} F_s J_s^2 \,\mathrm{d}^3 v_s. \tag{3.1}$$

Only the velocity volume of the support matters. In particular, if $A_c \subset \{E_s \le Kc\}$ for a fixed K, then

$$\|QE_{A_c}\| \le K_1 c^{3/4}, \qquad 0 < c \le 1. \tag{3.2}$$

To see this, $|v| \le K_2\sqrt{c}$, $0 < \mu \le K_3 c$, and the remaining factors in (3.1) are uniformly bounded. All constants below may depend on fixed coefficients and the chosen support but not on c or on the vector in the subspace.

3.2 Interior packets

Fix one species and write $m = m_s$. Set $B_* := \max_{\overline{I}} B$. Choose a rectangle $W = (q_0, q_1) \times (r_0, r_1) \subset (0,\infty)^2$ and use the invariants

$$\mu = cr, \qquad E = c(q + B_* r), \qquad v = \sqrt{c}\, w(\theta, q, r), \qquad w = \sqrt{\frac{2}{m}\{q + r(B_* - B(\theta))\}}. \tag{3.3}$$

The positive velocity branch is sufficient. The function w is bounded above and away from zero. Let $0 \neq \chi \in C_c^\infty(I)$ and define

$$N(q,r) = \int_I \frac{|\chi(\theta)|^2}{\beta(\theta) w(\theta, q, r)}\, \mathrm{d}\theta, \qquad \Phi_{\omega,c}(\theta, q, r) = \int_{\theta_0}^{\theta} \frac{\omega + \Omega(s, E, \mu)}{\beta(s) v(s, E, \mu)}\, \mathrm{d}s. \tag{3.4}$$

Here $\Omega(s, E, \mu)$ means the drift evaluated on the chosen velocity branch. The normalization N has positive upper and lower bounds on W.

Theorem 3.1 (Uniform subspace estimate)**.** *For every $\omega \in \mathbb{R}$ and $0 < c \le 1$, there is a closed infinite-dimensional subspace $\mathcal{W}_c^\omega \subset \mathcal{D}(L)$ such that*

$$\|(T_0 - \mathrm{i}\omega)u\| \le K_2\sqrt{c}\, \|u\|, \tag{3.5}$$

$$\|Qu\| \le K_1 c^{3/4} \|u\|, \tag{3.6}$$

$$\|(L - \mathrm{i}\omega)u\| \le \frac{1}{m_0}\left[K_2\sqrt{c} + K_1(\|G\| + |\omega|\|R\|)c^{3/4}\right] \|u\|. \tag{3.7}$$

Every vector in $\mathcal{W}_c^\omega$ is supported in $\{E \le Kc\}$ and in the interior of I. In particular, every bounded choice $u_c \in \mathcal{W}_c^\omega$ converges weakly to zero as $c \downarrow 0$.

Proof. For $g \in C_c^\infty(W)$, set all other species and the negative velocity branch to zero and define

$$(V_c g)(\theta, v, \mu) = \frac{m}{\sqrt{2\pi}} c^{-3/4} F(E)^{1/2} N(q,r)^{-1/2} \chi(\theta) g(q,r) e^{-\mathrm{i}\Phi_{\omega,c}(\theta,q,r)}. \tag{3.8}$$

The change of variables has

$$\mathrm{d}m_s = \frac{2\pi}{m_s^2}\, \mathrm{d}\tau\, \mathrm{d}E\, \mathrm{d}\mu, \qquad \mathrm{d}\tau = \frac{\mathrm{d}\theta}{\beta v}, \qquad \mathrm{d}m_s = \frac{2\pi}{m^2} c^{3/2} \frac{\mathrm{d}\theta\, \mathrm{d}q\, \mathrm{d}r}{\beta w} \tag{3.9}$$

on a positive branch. The factor m_s^{-2} follows from $\mathrm{d}v = \mathrm{d}E/(m_s v)$ and is needed when the species masses differ. Substitution in (2.2) proves $\|V_c g\| = \|g\|_{L^2(W)}$.

All factors other than χ and the phase are constant along characteristics. Moreover $X\Phi_{\omega,c} = \omega + \Omega$. Therefore

$$(T_0 - \mathrm{i}\omega) V_c g = -\frac{m}{\sqrt{2\pi}} c^{-3/4} F^{1/2} N^{-1/2} \beta v \chi' g e^{-\mathrm{i}\Phi_{\omega,c}}.$$

Its squared norm is

$$c \int_W \frac{|g|^2}{N(q,r)} \left(\int_I \beta w |\chi'|^2\, \mathrm{d}\theta \right) \mathrm{d}q\, \mathrm{d}r,$$

which proves (3.5). The support condition and (3.2) give (3.6). Finally,

$$(L - z)u = M^{-1}\big[(T_0 - z)u + (G + zR)Qu\big] \tag{3.10}$$

gives (3.7).

The transport estimate also bounds $T_0 V_c g$ for fixed c, ω. Since T_0 is closed, V_c extends from the dense smooth class to an isometry $L^2(W) \to \mathcal{D}(T_0)$. Its range is the required closed subspace. If E_c projects onto $\{E \le Kc\}$, then $E_c \to 0$ strongly by dominated convergence. Thus $|\langle f, u_c\rangle| \le \|E_c f\|\|u_c\| \to 0$. □

The theorem uses no bounce harmonics, no turning point, and no low-energy expansion of the drift. Large phase gradients at small c are allowed because the phase cancels exactly in the transport equation.

3.3 Outflow packets

The absorbing boundary allows a different envelope that is nonzero at outflow.

Theorem 3.2 (The open left half-plane)**.** *For the absorbing realization and every $z \in \mathbb{C}$ with $\Re z < 0$, there are closed infinite-dimensional subspaces $\mathcal{W}_c^z \subset \mathcal{D}(L)$ supported in $\{E \le Kc\}$ such that*

$$\|(L - z)|_{\mathcal{W}_c^z}\| \le \frac{1}{m_0}\left[K_z\sqrt{c}\, e^{-\kappa_z/\sqrt{c}} + K_1(\|G\| + |z|\|R\|)c^{3/4}\right]. \tag{3.11}$$

The constants are positive and independent of c for sufficiently small c.

Proof. Use (3.3). Let $\tau(\theta) = \int_a^\theta (\beta v)^{-1}\,\mathrm{d}s$ and $T = \tau(b)$. Write $z = -\alpha + \mathrm{i}\omega$, $\alpha > 0$. Choose a smooth η that vanishes near a and equals one on $[\theta_1, b]$, with $a < \theta_1 < b$. Let $\Psi' = \Omega$ in characteristic time. Define

$$H_{c,z}(\tau, q, r) = \eta(\theta(\tau))e^{-z(\tau - T) - \mathrm{i}(\Psi(\tau) - \Psi(T))}, \qquad N_{c,z}(q, r) = \int_0^T |H_{c,z}|^2\,\mathrm{d}\tau.$$

Uniformly on W, $T - \tau(\theta_1) \ge \kappa/\sqrt{c}$. Hence

$$\frac{1 - e^{-2\alpha\kappa/\sqrt{c}}}{2\alpha} \le N_{c,z} \le \frac{1}{2\alpha}.$$

The formula

$$(V_{c,z} g) = \frac{m}{\sqrt{2\pi}} c^{-1} F(E)^{1/2} N_{c,z}^{-1/2} H_{c,z}\, g(q, r)$$

is an isometry by (3.9). The inflow trace is zero. On each passing characteristic it belongs to the domain of the killed translation generator; the only residual under $T_0 - z$ is $-(\beta v)\eta'$ times the exponential. Since $|\beta v \eta'| \le K\sqrt{c}$ and its support precedes θ_1,

$$\int_0^{\tau(\theta_1)} e^{2\alpha(\tau - T)}\,\mathrm{d}\tau \le \frac{1}{2\alpha} e^{-2\alpha\kappa/\sqrt{c}}.$$

This proves the transport term in (3.11). Equation (3.2), followed by (3.10), proves the rest. Extension from smooth g to all of $L^2(W)$ again uses closedness. The same support argument proves weak nullity. □

4 Trapped orbits and the passing separatrix

The low-energy construction does not describe every source of long residence times. This section recovers two other families: oscillations on closed trapped orbits and passing particles at finite energy whose transit time diverges near a separatrix. They give different locations in phase space, although they lead to the same spectral inclusion.

4.1 Closed orbits and exact frequency attainment

Suppose an interior magnetic well has a regular trapped pitch interval Λ: for $\lambda \in \Lambda$, the component of $\{\theta : 1-\lambda B(\theta) > 0\}$ around the well is bounded by two simple turning points. Work on a compact subinterval $\Lambda_0 \Subset \Lambda$. Set $\mu = \lambda E$. On each orbit, travel time is a periodic coordinate and

$$T_s(E, \lambda E) = E^{-1/2}\mathcal{T}_s(\lambda), \qquad \mathcal{T}_s(\lambda) = \sqrt{2m_s}\int_{\theta_-(\lambda)}^{\theta_+(\lambda)} \frac{\mathrm{d}\theta}{\beta(\theta)\sqrt{1-\lambda B(\theta)}}. \tag{4.1}$$

Both endpoints are included through the integrable square-root singularities. On Λ_0, $\mathcal{T}_s$ is positive, continuous, and bounded above and below. These assertions follow by the substitution $\theta = \theta_- + (\theta_+ - \theta_-)\sin^2 u$, which removes the endpoint singularities. Formula (3.9) holds on both halves of the orbit and therefore across a full bounce.

Let $\overline{\Omega}_s(E, \lambda)$ be the time average of the drift. The periodic gauge

$$\psi_s(\tau) = \int_0^\tau (\Omega_s(r) - \overline{\Omega}_s)\,\mathrm{d}r$$

conjugates the fiber transport to $-\partial_\tau - \mathrm{i}\overline{\Omega}_s$. Thus its eigenvalues are $\mathrm{i}\omega_{s,n}$, where

$$\omega_{s,n}(E, \lambda) = \frac{2\pi n\sqrt{E}}{\mathcal{T}_s(\lambda)} - \overline{\Omega}_s(E, \lambda), \qquad n \in \mathbb{Z}. \tag{4.2}$$

The corresponding periodic eigenfunctions are $e^{-\mathrm{i}\psi_s(\tau)}e^{-2\pi\mathrm{i}n\tau/T_s}$. This is a direct-integral description of orbit transport, not a diagonalization of the field-coupled generator. Bounce and transit resonances also underlie orbit-based gyrokinetic eigenvalue methods [9].

Proposition 4.1 (Trapped singular subspaces)**.** *Assume a regular interior trapped family as above. For each $\omega \in \mathbb{R}$ there are closed infinite-dimensional subspaces $\mathcal{Z}_j \subset \mathcal{D}(L)$ supported on trapped orbits with $E \to 0$, such that*

$$\|(L - \mathrm{i}\omega)|_{\mathcal{Z}_j}\| \longrightarrow 0, \qquad \|Q|_{\mathcal{Z}_j}\| \longrightarrow 0.$$

Every bounded selection from these subspaces is weakly null. The result applies to absorption and to a periodic cell containing the well.

Proof. Fix an interior pitch λ_0 and energies $E_j \downarrow 0$. Choose $n_j \in \mathbb{Z}$ nearest to $(\omega + \overline{\Omega}_s(E_j, \lambda_0))T_s(E_j, \lambda_0 E_j)/(2\pi)$. The distance between the chosen fiber frequency and ω is at most $\pi\sqrt{E_j}/\mathcal{T}_s(\lambda_0)$. By continuity, choose an invariant open neighborhood W_j of (E_j, λ_0) with $E \le 2E_j$ on which the frequency error is at most twice this bound. Multiplying the normalized fiber eigenfunction by an arbitrary amplitude in $L^2(W_j)$ defines an isometry after the positive orbit weight $2\pi E T_s/(m_s^2 F_s)$ is absorbed into the amplitude. Its range is infinite-dimensional. Transport acts there by multiplication by $\mathrm{i}\omega_{s,n_j}$, so the residual has the asserted uniform bound.

The direct-integral generator domain contains this range because the frequencies are uniformly bounded on W_j.

The support estimate (3.2) bounds the field by $KE_j^{3/4}$. Equation (3.10) gives the coupled residual. The energy supports shrink to a null set, proving weak nullity. The orbits stay in the interior, so absorption does not change their evolution. □

The frequency lattice always gives approximation. Exact attainment on a prescribed pitch ray is a stronger assertion and depends on the energy dependence of the drift.

Proposition 4.2 (Homogeneous drift and a counterexample)**.** *Fix a regular pitch λ. If $\overline{\Omega}_s(E,\lambda) = b(\lambda)E$ for all $E > 0$, then*

$$\bigcup_{n\in\mathbb{Z}} \{\omega_{s,n}(E,\lambda) : E > 0\} = \mathbb{R}. \tag{4.3}$$

The conclusion can fail for a smooth nonhomogeneous drift, even though the closure of the union remains $\mathbb{R}$.

Proof. Write $a = 2\pi/\mathcal{T}_s(\lambda) > 0$ and $x = \sqrt{E}$. The frequencies are $nax - bx^2$. If $b = 0$, positive and negative n give the two half-lines, and $n = 0$ gives zero. If $b > 0$, the $n = 0$ branch gives every negative frequency. For $n > 0$, the maximum is $n^2a^2/(4b)$, which tends to infinity, and the branch crosses zero at $x = na/b > 0$. Hence all nonnegative frequencies are attained as well. Reversing signs handles $b < 0$.

For the counterexample, take $B = 2 - \cos\theta$, $\beta = 1$, a fixed trapped pitch, and $\Omega = aE/(1+E)$, independent of position. Then

$$\omega_n(E) = na\sqrt{E} - aE/(1+E).$$

For $n \le 0$ this is negative; for $n \ge 1$ it is positive, since $E/(1+E) < \sqrt{E}$. Thus zero is not attained at any positive energy. It is approached as $E \downarrow 0$. The lattice argument in the preceding proof still gives closure coverage of every real frequency. □

The homogeneous condition holds whenever the drift is linear in E and μ at fixed position. It is not a consequence of smoothness alone.

Remark 4.3 (Periodic passing fibers at fixed excess energy). For a periodic passing orbit, (4.2) holds with the transit period in place of the bounce period. When the drift is homogeneous, varying E at fixed pitch again covers every real frequency. This does not prove coverage at each fixed excess $c_* = E - \mu B_* > 0$, because that variation also changes c_*. For example, when the orbit-averaged drift is zero, the transit bound in Proposition 6.1 gives

$$\omega_n = \frac{2\pi n}{T}, \qquad |\omega_n| \ge \frac{2\pi}{T_*(c_*)} \quad (n \ne 0),$$

whereas $\omega_0 = 0$. Thus the nonzero fiber frequencies have a gap at fixed positive excess, even though the union over all excess energies covers the axis. No fixed-excess surjectivity is used in the spectral theorems here.

4.2 Passing particles at finite energy

Assume B has a nondegenerate maximum B_* at a point in $\overline{I}$. For a boundary maximum, assume a smooth extension with zero first derivative and negative second derivative there. Choose $0 < \mu_0 < \mu_1$ and define

$$A_\delta = \{\mu_0 < \mu < \mu_1,\ \delta < E - \mu B_* < 2\delta,\ v > 0\}.$$

A particle in this set has energy bounded away from zero. Its full transit time is

$$T(E,\mu) = \sqrt{m_s/2} \int_a^b \frac{\mathrm{d}\theta}{\beta(\theta)\sqrt{E - \mu B_* + \mu(B_* - B(\theta))}}. \tag{4.4}$$

Taylor bounds on $B_* - B$ near the maximum give, uniformly on A_δ, $T(E,\mu) \geq c_1 \log(1/\delta)$ for sufficiently small δ. An upper bound of the same order holds when all maxima at height B_* are finitely many and nondegenerate. For the construction below only the lower bound is needed.

Theorem 4.4 (Finite-energy passing subspaces)**.** *For absorption and every fixed $\omega \in \mathbb{R}$, there are closed infinite-dimensional subspaces $\mathcal{Z}_\delta^\omega \subset \mathcal{D}(L)$ supported in A_δ such that*

$$\|(L - \mathrm{i}\omega)|_{\mathcal{Z}_\delta^\omega}\| \leq K_\omega \left\{ \frac{1}{\log(1/\delta)} + \delta^{1/4} \right\}, \qquad \|Q|_{\mathcal{Z}_\delta^\omega}\| \leq K\delta^{1/4}. \tag{4.5}$$

All bounded selections are weakly null as $\delta \downarrow 0$. After cutting the cell, the construction also gives interior-supported quasimodes on a periodic domain; it does not prescribe an exact frequency on an individual periodic orbit.

Proof. On each positive passing orbit let $\tau \in (0,T)$ be travel time from inflow. Choose a nonzero $\chi \in C_c^\infty(0,1)$ and use the fiber profile

$$\frac{\chi(\tau/T)}{\sqrt{T}\,\|\chi\|_{L^2(0,1)}} \exp\left[-\mathrm{i}\omega\tau - \mathrm{i}\int_0^\tau \Omega(r)\,\mathrm{d}r \right].$$

It has unit norm in $L^2(0,T)$, zero trace at both ends, and transport residual $\|\chi'\|/(T\|\chi\|)$. Multiply by arbitrary square-integrable invariant amplitudes, absorbing the weight $2\pi/(m_s^2 F_s)$ into their normalization. The lower bound on T gives the first transport estimate uniformly on the whole resulting subspace. The domain assertion follows by closedness from smooth amplitudes compactly supported in the invariant window.

At each position the velocity support is contained in the shell $0 < E - \mu B_* < 2\delta$. On each half-line its length at fixed μ is at most $\sqrt{4\delta/m_s}$. Maxwellian-weighted integration in μ and (3.1) therefore give the second estimate in (4.5). Equation (3.10) gives the coupled bound. The supports shrink to the separatrix, a phase-space null set, which proves weak nullity. □

These packets show why low total energy is sufficient but not necessary for slow passing dynamics. The converse in Section 6 locates passing sequences by excess energy, which includes both mechanisms.

5 Essential spectra and reductions

For a closed densely defined operator A, write $z \in \sigma_{\mathrm{sap}}(A)$ when there exist $u_n \in \mathcal{D}(A)$ with

$$\|u_n\| = 1, \qquad u_n \rightharpoonup 0, \qquad (A - z)u_n \to 0. \tag{5.1}$$

This is the singular approximate point spectrum. We use the Fredholm essential spectrum $\sigma_{\mathrm{ess}}(A) = \{z : A - z \text{ is not Fredholm}\}$. These conventions avoid identifying the different essential spectra available for nonnormal operators.

Lemma 5.1. *A sequence satisfying* (5.1) *prevents $A - z$ from having both closed range and finite-dimensional kernel. Thus $\sigma_{\mathrm{sap}}(A) \subset \sigma_{\mathrm{ess}}(A) \subset \sigma(A)$. It remains a singular sequence after any bounded compact perturbation of A.*

Proof. If the range were closed and the kernel finite-dimensional, restriction to the orthogonal complement of the kernel would have a bounded inverse onto the range. Consequently $\operatorname{dist}(u, \ker(A - z)) \le K\|(A - z)u\|$ for $u \in \mathcal{D}(A)$. The distance for u_n tends to one, contradicting the residual. A compact operator sends a bounded weakly null sequence to a norm-null sequence. □

Theorem 5.2 (Exact boundary spectra)**.** *Under the assumptions of Section 2, the following hold.*

(i) For every closed interior realization, $\mathrm{i}\mathbb{R} \subset \sigma_{\mathrm{sap}}(L)$.

(ii) For the absorbing realization, $\{\Re z \le 0\} \subset \sigma_{\mathrm{sap}}(L)$, with arbitrary drive satisfying (2.6)*.*

(iii) If $G = 0$, the absorbing realization satisfies

$$\sigma(L) = \sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{sap}}(L) = \{z : \Re z \le 0\}. \tag{5.2}$$

For periodic boundaries,

$$\sigma(L) = \sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{sap}}(L) = \mathrm{i}\mathbb{R}. \tag{5.3}$$

Proof. Choose a unit vector from each subspace in Theorems 3.1 and 3.2, and let $c \downarrow 0$. Their support and residual estimates give (i) and (ii). For $G = 0$, Proposition 2.1 excludes the open right half-plane in the absorbing case and the complement of the axis in the periodic case. Lemma 5.1 gives both equalities. □

Theorem 5.3 (Finite-codimension compression)**.** *Let $\Sigma = \mathrm{i}\mathbb{R}$ for a closed interior realization and $\Sigma = \{\Re z \le 0\}$ for absorption. Let $V \subset \mathcal{H}$ be closed with finite codimension, let Π_V be any bounded projection onto V, and let K be bounded and compact. Define*

$$A_V u = \Pi_V (L + K)u, \qquad u \in \mathcal{D}(L) \cap V. \tag{5.4}$$

For every $z \in \Sigma$, this domain contains unit vectors $u_n \rightharpoonup 0$ in V with $(A_V - z)u_n \to 0$. Thus $\Sigma \subset \sigma_{\mathrm{sap}}(A_V)$ whenever A_V is closed. The same inclusion holds for any closed extension on V.

For every c, the vectors may be chosen in the infinite-dimensional space $\mathcal{W}_c^z \cap V$, without a codimension-dependent loss in the residual estimate. The conclusions also hold after bounded similarity.

Proof. The map $\mathcal{W}_c^z \to \mathcal{H}/V$ has finite-dimensional range, so its kernel $\mathcal{W}_c^z \cap V$ is infinite-dimensional. Choose a unit vector u_c there. Since $\Pi_V u_c = u_c$,

$$\|(A_V - z)u_c\| \le \|\Pi_V\|\big(\|(L - z)u_c\| + \|K u_c\|\big) \longrightarrow 0.$$

In fact $\|K|_{\mathcal{W}_c^z}\| \to 0$ uniformly: the projections E_c onto the shrinking energy support satisfy $E_c \to 0$ strongly, so compactness gives $\|KE_c\| \to 0$. All chosen vectors are weakly null.

A bounded invertible map sends weakly null vectors to weakly null vectors and changes normalized residuals by at most its condition number. It also preserves finite codimension and compactness. This proves the similarity statement. Closedness of the compression is not automatic and is not needed for the displayed residual conclusion. □

A generator extending (5.4) cannot have a uniformly exponentially decaying semigroup, since its resolvent would exist on the imaginary axis. Repeating the argument also permits any finite composition of the stated operations, provided each operator whose spectrum is invoked is closed.

Corollary 5.4 (Local extension to the line)**.** *Suppose the model is defined on the whole line, R, G, Q, M^{-1} are globally bounded, and the transport realization is closed and contains compactly supported interior functions. If one finite open interval has the local coefficient properties of Section 2, then $\mathrm{i}\mathbb{R} \subset \sigma_{\mathrm{sap}}(L)$, for any finite number of kinetic species. The compression conclusions of Theorem 5.3 remain valid.*

Proof. Choose χ inside that interval and use its local maximum of B in (3.3). The packet is zero outside the interval. The proof of Theorem 3.1 is unchanged; global boundedness is used only in (3.10). □

Corollary 5.5 (Mirror-free reference transport)**.** *Replace the transport by $-v\beta(\theta)\partial_\theta - \mathrm{i}\Omega$, and use throughout a fixed reference field $B_{\mathrm{ref}} > 0$, the Maxwellian $F_{\mathrm{ref}} = (m_s/2\pi)^{3/2} e^{-m_s v^2/2 - \mu B_{\mathrm{ref}}}$, the velocity measure $2\pi B_{\mathrm{ref}}\,\mathrm{d}v\,\mathrm{d}\mu/m_s$, and $\sigma_{\mathrm{ref}} = (B_{\mathrm{ref}}\beta)^{-1}$. Under the corresponding boundedness and positivity assumptions, Theorems 3.1, 3.2, 5.2, and 5.3 hold for this reference model.*

Proof. The reference model is precisely the constant-B instance of the proofs, with arbitrary allowed gyroaverage, drift, and drive coefficients. Its flow preserves its reference Maxwellian and measure. The same changes of variables and field-volume estimate apply. No claim is made for deleting the mirror force while retaining a noninvariant variable-B Maxwellian. □

6 A converse for passing quasimodes

This section concerns absorption. Let $B_* := \max_{\overline{I}} B$ and set

$$\mathcal{P}_s = \{E_s > \mu B_*\}, \qquad \mathcal{P}_{s,c} = \{E_s - \mu B_* \geq c\}, \qquad c > 0. \tag{6.1}$$

Let E_{p} and E_c^+ be the corresponding projections on $\mathcal{H}$. These are invariant sets for transport. Their indicator projections commute with $U_0(t)$ and preserve $\mathcal{D}(T_0)$. They need not commute with L.

6.1 The transport cutoff

Proposition 6.1 (Exact extinction time)**.** *The transport semigroup restricted to* $\operatorname{ran} E_c^+$ *vanishes for $t \geq T_*(c)$, where*

$$T_*(c) = \max_s \sqrt{\frac{m_s}{2c}} \int_a^b \frac{\mathrm{d}\theta}{\beta(\theta)}. \tag{6.2}$$

This is the smallest such time. The restricted generator $T_{0,c}$ has empty spectrum. In particular,

$$\|(T_{0,c} - \mathrm{i}\omega)^{-1}\| = \frac{2T_*(c)}{\pi}, \qquad \inf_{0 \neq u \in \mathcal{D}(T_{0,c})} \frac{\|(T_{0,c} - \mathrm{i}\omega)u\|}{\|u\|} = \frac{\pi}{2T_*(c)}. \tag{6.3}$$

Proof. Every trajectory in the cutoff has $|v| \geq \sqrt{2c/m_s}$ and crosses the interval monotonically. Its crossing time is at most the corresponding term in (6.2). As $\mu \downarrow 0$ and $E_s - \mu B_* \downarrow c$, the full crossing times approach that term. For every smaller time, a positive-measure set of trajectories starting near inflow is still inside, so the semigroup is nonzero.

For any $z \in \mathbb{C}$, nilpotency makes

$$(z - T_{0,c})^{-1} = \int_0^{T_*(c)} e^{-zt} U_{0,c}(t)\,\mathrm{d}t$$

a bounded two-sided inverse. This follows by differentiating on the generator domain and then using density. At $z = \mathrm{i}\omega$, multiplication by the unit-modulus characteristic phase conjugates each fiber inverse to the Volterra integration operator on $(0, T(E,\mu))$. Its norm is $2T/\pi$: the sharp inequality $\|f\|_{L^2(0,T)} \le (2T/\pi)\|f'\|_{L^2(0,T)}$ for $f(0) = 0$ follows from the first eigenvalue $(\pi/2T)^2$ of $-\partial_\tau^2$ with $f(0) = f'(T) = 0$. Equality is attained by $\sin(\pi\tau/(2T))$. The essential supremum of the fiber lengths is $T_*(c)$, including positive-measure sets arbitrarily close to it. Taking the direct-integral norm proves both equalities in (6.3). □

This is a statement about the restricted transport operator. The full field-coupled dynamics generally does not preserve the cutoff, so it has no corresponding restriction without an additional projection.

Corollary 6.2 (No unitary intertwining of passing transport)**.** *Let $U_{0,\mathrm{p}}(t)$ denote absorbing transport on the full passing cone. If $V(t)$ is a unitary group and a bounded map J satisfies $JU_{0,\mathrm{p}}(t) = V(t)J$ for all $t \ge 0$, then $J = 0$.*

Proof. For $u \in \operatorname{ran} E_c^+$, extinction gives $V(T_*(c))Ju = 0$, hence $Ju = 0$. The union of these cutoff subspaces is dense in the passing space. Boundedness completes the argument. □

Lemma 6.3 (Volume of a low-energy passing region)**.** *There is K independent of $0 < c \le 1$ such that*

$$\|Q(E_{\mathrm{p}} - E_c^+)\| \le K c^{1/4}. \tag{6.4}$$

Proof. At fixed θ, μ, each velocity half-line in the shell has length at most $\sqrt{2c/m_s}$, since

$$|v| \in \left(\sqrt{\frac{2\mu(B_* - B)}{m_s}}, \sqrt{\frac{2\{\mu(B_* - B) + c\}}{m_s}} \right).$$

Using $F_s \le (m_s/2\pi)^{3/2} e^{-\mu B(\theta)}$ and integrating in μ bounds the weighted shell volume by $K\sqrt{c}$, uniformly in θ. Equation (3.1) completes the proof. □

6.2 Compactness after transport

Define the bounded source map

$$S_\omega = G + \mathrm{i}\omega R, \qquad (S_\omega \phi)_s = \mathrm{i}(d_s + \omega q_s) F_s J_s \phi. \tag{6.5}$$

Lemma 6.4 (Compactness of the resolvent contribution)**.** *For every fixed $c > 0$ and $\omega \in \mathbb{R}$, the operator*

$$K_{c,\omega} = Q(T_{0,c} - \mathrm{i}\omega)^{-1} E_c^+ S_\omega : \mathcal{X} \longrightarrow \mathcal{X} \tag{6.6}$$

is compact. More precisely, before applying the bounded factor C^{-1}, it is an integral operator with a bounded kernel, hence is Hilbert–Schmidt.

Proof. On the positive branch, the inverse is a Volterra integral along the characteristic from a to θ. The phase has modulus one. Substitution in R^* gives, up to a constant complex factor, a kernel with absolute value bounded by

$$\frac{2\pi|q_s|B(\theta)}{m_s^2\beta(\theta')}\int_{E-\mu B_*\geq c}\frac{F_s(E)|d_s(\theta',E,\mu)+\omega q_s|}{v_s(\theta,E,\mu)v_s(\theta',E,\mu)}\,\mathrm{d}E\,\mathrm{d}\mu. \tag{6.7}$$

The actual kernel also contains the indicator $\theta'<\theta$ and the two gyroaverage factors, each bounded by one. On the negative branch the ordering is reversed, with absolute speeds in the denominator.

For fixed θ', convert the integral in (6.7) to $\mathrm{d}^3v_s(\theta')$. One denominator is absorbed by this measure and the other is at least $\sqrt{2c/m_s}$. Cauchy–Schwarz and (2.6) therefore give a bound $K_c(1+|\omega|)$ independent of θ,θ'. There are finitely many species and two signs. Since σ is bounded above and below and I is finite, the kernel is square-integrable for the field measure. Composition with C^{-1} preserves compactness. □

The estimate is deliberately at fixed c. Its constant can diverge as $c\downarrow 0$; no uniform compactness is used.

6.3 Localization of every passing sequence

The complementary region detects the field through

$$\kappa_\omega(\theta)=\sum_s\int_{E_s\leq\mu B_*}|d_s+\omega q_s|^2F_sJ_s^2\,\mathrm{d}^3v_s. \tag{6.8}$$

It is enough that this function be positive almost everywhere; a uniform lower bound is not required.

Theorem 6.5 (Passing quasimodes have vanishing field)**.** *Fix $\omega\in\mathbb{R}$. Suppose either $\kappa_\omega>0$ almost everywhere on I, or $S_\omega=0$. If*

$$u_n\in\mathcal{D}(L),\quad \|(\mathrm{Id}-E_{\mathrm{p}})u_n\|\to 0,\quad \|u_n\|=1,\quad r_n:=(L-\mathrm{i}\omega)u_n\to 0, \tag{6.9}$$

then

$$Qu_n\longrightarrow 0\quad \textit{in } \mathcal{X},\qquad \|E_c^+u_n\|\longrightarrow 0\quad \textit{for every fixed } c>0. \tag{6.10}$$

Thus every asymptotically passing quasimode sequence concentrates in arbitrarily thin excess-energy shells and is weakly null. In particular, no normalized sequence supported above a fixed positive excess energy can have vanishing residual on the imaginary axis.

Proof. Write $\phi_n=Qu_n$. It is bounded in $\mathcal{X}$. By (3.10),

$$Mr_n=(T_0-\mathrm{i}\omega)u_n+S_\omega\phi_n. \tag{6.11}$$

Put $f_n=(\mathrm{Id}-E_{\mathrm{p}})u_n$, so $f_n\to 0$. If $f_n=0$, projecting (6.11) onto the complementary region gives

$$\int_I\sigma\kappa_\omega|\phi_n|^2\,\mathrm{d}\theta\leq\|Mr_n\|^2\longrightarrow 0.$$

For the stated, asymptotically passing hypothesis, use the adjoint domain instead. For every $g\in\mathcal{D}(T_0^*)\cap\mathrm{ran}(\mathrm{Id}-E_{\mathrm{p}})$,

$$\langle g,S_\omega\phi_n\rangle=\langle g,Mr_n\rangle-\langle(T_0-\mathrm{i}\omega)^*g,f_n\rangle\longrightarrow 0.$$

The complementary projection commutes with the transport semigroup, so this test domain is dense in its range. Every weak cluster point ϕ of the bounded fields therefore satisfies $(\mathrm{Id} - E_{\mathrm{p}})S_\omega\phi = 0$. If $\kappa_\omega > 0$ almost everywhere, its squared norm is $\int \sigma\kappa_\omega|\phi|^2$, forcing $\phi = 0$. Thus $\phi_n \rightharpoonup 0$.

At fixed $c > 0$, invert the transport part of (6.11):

$$E_c^+ u_n = (T_{0,c} - \mathrm{i}\omega)^{-1} E_c^+ M r_n - (T_{0,c} - \mathrm{i}\omega)^{-1} E_c^+ S_\omega \phi_n. \tag{6.12}$$

Applying Q and Lemma 6.4 gives $QE_c^+ u_n \to 0$: the first term tends to zero by (6.3), and the compact second term sends weakly null fields to zero. Since $Qf_n \to 0$, Lemma 6.3 now gives

$$\limsup_n \|\phi_n\| \le K c^{1/4}.$$

Letting $c \downarrow 0$ proves strong field convergence. Equation (6.12) then proves the second conclusion. If $S_\omega = 0$, equation (6.12) directly gives $E_c^+ u_n \to 0$, and the shell estimate gives $\phi_n \to 0$ without the complementary-region argument. Finally, for fixed $g \in \mathcal{H}$,

$$|\langle g, u_n\rangle| \le \|g\|(\|f_n\| + \|E_c^+ u_n\|) + \|(E_{\mathrm{p}} - E_c^+)g\|.$$

Take $n \to \infty$ and then $c \downarrow 0$ to obtain weak nullity. □

Corollary 6.6 (An elementary coefficient test)**.** *Suppose $B(\theta) < B_*$ for almost every θ, $J_s \neq 0$ almost everywhere in velocity, and $d_s(\theta, E, \mu) = a_s(\theta) + b_s(\theta)E$. If, for almost every θ, at least one species has $(a_s(\theta) + \omega q_s, b_s(\theta)) \neq (0, 0)$, then $\kappa_\omega > 0$ almost everywhere. In particular, if the affine drive coefficients are independent of θ, then for each ω either this condition holds or $S_\omega = 0$.*

Proof. At every position with $B < B_*$, the complementary region contains an open set of velocity pairs. A nonzero affine polynomial in E cannot vanish on that set except on a null subset. The Maxwellian is positive and the gyroaverage is nonzero almost everywhere, so its contribution to (6.8) is positive. □

For constant B, the complementary region has zero measure and this coefficient test is unavailable. The existence theorems and the exact boundary spectra still apply. The converse is not asserted there for a general nonzero S_ω.

7 Velocity diffusion at small collision frequency

A velocity-differential collision operator is unbounded on the kinetic energy space. The collisionless singular sequences therefore do not survive merely by a bounded-perturbation argument. Their derivatives must be estimated before the collision frequency is sent to zero.

For definiteness, consider the Maxwellian Ornstein–Uhlenbeck operator, written in the gyrotropic variables of Section 2:

$$(\mathcal{C}h)_s = \partial_v[F_s \partial_v(h_s/F_s)] + \frac{2m_s}{B}\partial_\mu[\mu F_s \partial_\mu(h_s/F_s)]\,. \tag{7.1}$$

At fixed position this is $\Delta_{\boldsymbol{v}} h_s + m_s \nabla_{\boldsymbol{v}} \cdot (\boldsymbol{v} h_s)$ on gyrotropic functions. On compactly supported smooth functions its quadratic form is

$$\langle h, \mathcal{C}h\rangle = -\sum_s \int \sigma \,\mathrm{d}\theta \,\mathrm{d}^3 v_s \, F_s \left[|\partial_v(h_s/F_s)|^2 + \frac{2m_s\mu}{B}|\partial_\mu(h_s/F_s)|^2\right].$$

This model fixes a Maxwellian bath. It does not conserve the perturbed momentum and energy as the full linearized Landau operator does. The distinction matters when comparing weak-collision kinetic spectra [1, 10, 13].

Let L_ν be a closed realization whose domain contains the smooth interior functions constructed below and whose action on them is

$$L_\nu = M^{-1}(T_0 + GQ + \nu\mathcal{C}), \qquad 0 < \nu \le 1. \tag{7.2}$$

For the following theorem no generation assertion is needed. Statements about its semigroup are made only when the selected realization generates one. Suppose, on the selected species and on the packet support, that

$$\Omega(\theta, E, \mu) = Ea(\theta, E, \mu) + \mu b(\theta, E, \mu), \tag{7.3}$$

where a, b and their derivatives through order two are bounded near $(E, \mu) = (0, 0)$. This includes smooth drifts linear in the two energy variables. No velocity derivatives of the drive are required.

Lemma 7.1 (Diffusion estimate for localized states)**.** *For every $W_0 < \infty$ and every positive integer d, the spaces $\mathcal{W}_c^\omega$ in Theorem 3.1 contain d-dimensional subspaces $\mathcal{E}_{c,d}^\omega$ of smooth interior functions such that, for $|\omega| \le W_0$,*

$$\|\mathcal{C}u\| \le K_d(c^{-1} + \omega^2 c^{-2})\|u\|, \qquad u \in \mathcal{E}_{c,d}^\omega. \tag{7.4}$$

The constant is independent of c and ω in the displayed range.

Proof. Choose a fixed d-dimensional space of smooth amplitudes supported in a smaller rectangle $W' \Subset W$ in (3.3). Its derivatives through order two are bounded in L^2 by a constant times its L^2 norm. Use its image under the isometry (3.8).

On the rescaled support, the phase has the form

$$\Phi_{\omega,c}(\theta, q, r) = \omega c^{-1/2}\varphi(\theta, q, r) + c^{1/2}\psi_c(\theta, q, r), \tag{7.5}$$

where φ, ψ_c, and their invariant derivatives through order two are uniformly bounded. This follows directly by inserting (7.3) into (3.4); w is bounded away from zero. At fixed θ,

$$q = \frac{mv^2/2 + \mu(B - B_*)}{c}, \qquad r = \frac{\mu}{c},$$

so one v derivative costs $O(c^{-1/2})$ on amplitudes and $O(|\omega|c^{-1} + 1)$ on the phase. One μ derivative costs $O(c^{-1})$ on amplitudes and $O(|\omega|c^{-3/2} + c^{-1/2})$ on the phase. The collision operator expands as

$$\mathcal{C}u = \partial_v^2 u + mv\partial_v u + mu + \frac{2m}{B}\{\mu\partial_\mu^2 u + (1 + B\mu)\partial_\mu u + Bu\}.$$

Since $v = O(\sqrt{c})$ and $\mu = O(c)$, the second derivatives cost at most $K_d(c^{-1} + |\omega|c^{-3/2} + \omega^2 c^{-2})$ in norm. The derivatives of $F^{1/2}$ and $N^{-1/2}$ satisfy the same or smaller bounds. Finite-dimensional norm equivalence for the amplitudes makes these estimates uniform over the subspace. Finally $|\omega|c^{-3/2} \le \frac{1}{2}(c^{-1} + \omega^2 c^{-2})$ proves (7.4). □

7.1 A one-species linearized Landau model

The derivative estimate also applies to the one-species, unit-mass linearized Landau operator below and to a specified gyrophase compression after velocity-phase conjugation. Treating the nonlocal term directly avoids assuming that it is bounded.

For one unit-mass species, write the three-dimensional velocity as ξ, with $\mu = |\xi_\perp|^2/(2B)$, and let $F(\xi) = (2\pi)^{-3/2} e^{-|\xi|^2/2}$. Define

$$a_{ij}(z) = \frac{1}{|z|}\left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right), \tag{7.6}$$

$$\mathscr{Q}(f,g) = \partial_i \int_{\mathbb{R}^3} a_{ij}(\xi - \xi')\{f(\xi')\partial_j g(\xi) - g(\xi)\partial'_j f(\xi')\}\,\mathrm{d}\xi',$$

$$\mathscr{L}h = \mathscr{Q}(F,h) + \mathscr{Q}(h,F). \tag{7.7}$$

Repeated Cartesian indices are summed. Constants setting the collision frequency are absorbed into ν. The operator is applied locally at each position. It preserves gyrotropic functions because the background is isotropic.

To include a fixed perpendicular Fourier phase, let $U_\kappa h = e^{\mathrm{i}\kappa(\theta)\cdot\xi} h$, where κ is perpendicular to the field and uniformly bounded, and let P_{gy} be orthogonal gyrophase averaging. On the gyrotropic space set

$$\mathcal{C}_\kappa = P_{\mathrm{gy}} U_\kappa^{-1} \mathscr{L} U_\kappa P_{\mathrm{gy}}. \tag{7.8}$$

The case $\kappa = 0$ is the local linearized Landau operator. Formula (7.8) is a compression, not a unitary equivalence to the conservative operator. A multispecies gyrocenter collision model requires additional ordering and conservation choices [10, 14].

Proposition 7.2 (Estimate for Coulomb collisions)**.** *On the localized subspaces of Lemma 7.1,*

$$\|\mathcal{C}_\kappa u\| \le K_{d,\kappa}(c^{-1} + \omega^2 c^{-2})\|u\|, \tag{7.9}$$

with constants uniform for bounded ω and bounded κ. Consequently the residual and feedback estimates below hold for any closed realization that contains the constructed functions and acts on them with $\mathcal{C}$ replaced by $\mathcal{C}_\kappa$.

Proof. First, the full Cartesian Hessian of a packet obeys

$$\|\nabla_\xi^2 u\| \le K_d(c^{-1} + |\omega| c^{-3/2} + \omega^2 c^{-2})\|u\|.$$

Indeed, on its support $\xi = \sqrt{c}\,\zeta$, where ζ lies in a fixed compact set away from the parallel and perpendicular axes used in the chart. Its amplitude is smooth in ζ, while its phase has the form (7.5). Two Cartesian derivatives then give exactly the three displayed costs. The middle term is absorbed as in Lemma 7.1. First derivatives satisfy the corresponding smaller bound.

We next control the nonlocal velocity term. Integration by parts in (7.7), using $\partial_i\partial_j a_{ij} = -8\pi\delta_0$ in distributions, gives

$$\mathscr{L}h = (a * F)_{ij}\partial_i\partial_j h + (a * h)_{ij}\partial_i\partial_j F + 16\pi F h. \tag{7.10}$$

The coefficients $a * F$ are bounded. Conjugating the middle term from $L^2(F^{-1}\,\mathrm{d}\xi)$ to ordinary L^2 gives kernels

$$K_{ij}(\xi,\xi') = \frac{\partial_i\partial_j F(\xi)}{\sqrt{F(\xi)}} a_{ij}(\xi - \xi')\sqrt{F(\xi')}.$$

They are Hilbert–Schmidt: their squares are bounded by a polynomial times $F(\xi)F(\xi')|\xi-\xi'|^{-2}$, integrable at the diagonal in three dimensions and at infinity by Gaussian decay. The last term in (7.10) is bounded multiplication. Hence

$$\|\mathscr{L}h\|_{L^2(F^{-1})} \le C\{\|\nabla_\xi^2 h\|_{L^2(F^{-1})} + \|h\|_{L^2(F^{-1})}\}.$$

Multiplication by U_κ is unitary and adds only bounded multiples of the first derivatives and the function itself to its Hessian. Gyrophase averaging is an orthogonal contraction. Combining these facts and integrating in position proves (7.9). □

For $\kappa = 0$, the linearized operator has the entropy form

$$\langle h, \mathscr{L}h\rangle_{F^{-1}} = -\frac{1}{2}\iint FF' a_{ij}(\xi-\xi')(\partial_i f - \partial_i' f')\,\overline{(\partial_j f - \partial_j' f')}\,\mathrm{d}\xi\,\mathrm{d}\xi', \qquad f = h/F.$$

It is therefore dissipative and has the collision invariants F, $\xi_j F$, and $(|\xi|^2-3)F$ in its kernel. Conjugation and orthogonal compression preserve dissipativity in (7.8). The estimates do not assert that the finite-wavenumber compressed operator has the same kernel. They also do not establish generation for an arbitrary sum of transport and collisions; the realization and semigroup qualifications in (7.2) remain in force.

Theorem 7.3 (Small-collision pseudospectral bounds)**.** *Use either* (7.1) *or the operator in Proposition 7.2. Fix $W_0 < \infty$ and a nonnegative integer m. There are K_m and $\nu_0 > 0$ such that, for every closed subspace $V \subset \mathcal{H}$ of codimension at most m, every $|\omega| \le W_0$, and $0 < \nu < \nu_0$, one can choose a unit vector $u_{\nu,\omega} \in V \cap \mathcal{D}(L_\nu)$ with*

$$\|(L_\nu - \mathrm{i}\omega)u_{\nu,\omega}\| \le K_m\left(\nu^{1/3} + \nu^{1/5}|\omega|^{2/5}\right). \tag{7.11}$$

These vectors are supported at energies tending to zero, and their fields tend to zero. Consequently, whenever $\mathrm{i}\omega \in \rho(L_\nu)$,

$$\|(L_\nu - \mathrm{i}\omega)^{-1}\| \ge \frac{1}{K_0(\nu^{1/3} + \nu^{1/5}|\omega|^{2/5})}. \tag{7.12}$$

The same residual conclusion holds after a bounded projection onto V, with the constant multiplied by the projection norm.

Proof. Intersect V with $\mathcal{E}^{\omega}_{c,m+1}$ from Lemma 7.1. A dimension count gives a nonzero vector in the intersection. The estimates for the entire finite-dimensional space survive normalization. Writing $x = \sqrt{c}$, (3.10) and (7.4) give

$$\|(L_\nu - \mathrm{i}\omega)u\| \le K_m\{x + (1+|\omega|)x^{3/2} + \nu x^{-2} + \nu\omega^2 x^{-4}\}. \tag{7.13}$$

Set $x = \nu^{1/3} + (\nu\omega^2)^{1/5}$. For bounded ω this is at most one when ν is sufficiently small. Also $\nu x^{-2} \le \nu^{1/3}$ and $\nu\omega^2 x^{-4} \le (\nu\omega^2)^{1/5}$. Thus every term is bounded by a fixed multiple of x, proving (7.11). The support and field conclusions follow from Theorem 3.1. Applying the inverse to the residual gives (7.12). For a projection Π_V, use $\Pi_V u = u$ as in Theorem 5.3. □

Corollary 7.4 (A limit on uniform relaxation estimates)**.** *Suppose a generating realization, or a generating finite-codimension compression as above, satisfies*

$$\|S_\nu(t)\| \le M_* e^{-g_\nu t}, \qquad t \ge 0,$$

with M_* independent of ν. Then $g_\nu \le KM_*\nu^{1/3}$. For a contraction realization, the vectors at $\omega = 0$ also satisfy

$$\|S_\nu(t)u_{\nu,0}\| \ge 1 - Kt\nu^{1/3}.$$

These statements hold in the chosen norm; equivalent fixed norms only change the constants.

Proof. Exponential stability gives $\|L_\nu^{-1}\| \le M_*/g_\nu$ by its Laplace integral. Combine this with (7.12) at zero. The contraction estimate follows from $S_\nu(t)u - u = \int_0^t S_\nu(s)L_\nu u\,\mathrm{d}s$. □

The $\nu^{1/3}$ scale already occurs in weakly collisional phase mixing [1, 2]. Here it is a lower obstruction to uniform relaxation, obtained from packets on a bounded field line with the field constraint retained. The frequency-dependent $\nu^{1/5}|\omega|^{2/5}$ estimate accounts for the extra velocity derivatives of the characteristic phase. Neither exponent is asserted to be the sharp spectral gap of L_ν, and a pseudospectral bound does not locate a collisional eigenvalue. The constants depend on codimension because velocity derivatives of the amplitude space are now involved.

7.2 How large field feedback must become

The packet estimates also constrain attempts to accelerate weak collisional relaxation through the electrostatic field. A dynamic controller need not be finite dimensional for the following argument.

Theorem 7.5 (Necessary observation gain for uniform stabilization)**.** *Let $\mathcal{Y}$ be a fixed controller Hilbert space, and consider generating realizations on $\mathcal{H} \oplus \mathcal{Y}$ of*

$$\mathbb{A}_\nu = \begin{pmatrix} L_\nu + K_\nu Q & B_\nu \\ C_\nu Q & D_\nu \end{pmatrix}, \qquad G_\nu = 1 + \|K_\nu\| + \|C_\nu\|. \tag{7.14}$$

Here the four controller operators are bounded for each ν, and the realization contains the interior packet vectors $(u, 0)$. All norms use fixed, ν-independent inner products. If

$$\|e^{t\mathbb{A}_\nu}\| \le M_* e^{-g_\nu t}, \qquad t \ge 0, \tag{7.15}$$

with M_* independent of ν, then

$$g_\nu \le CM_* \left(\nu^{1/3} + \nu^{3/7} G_\nu^{4/7}\right). \tag{7.16}$$

In particular, a positive decay rate bounded below uniformly in ν requires $G_\nu \ge c\nu^{-3/4}$ for all sufficiently small ν. This is a necessary condition, not a stabilization construction.

Proof. Use the zero-frequency packet from Lemma 7.1 with a fixed smooth unit amplitude. The vector $(u_c, 0)$ has unit norm and satisfies

$$\|\mathbb{A}_\nu(u_c, 0)\| \le C\{c^{1/2} + \nu c^{-1} + G_\nu c^{3/4}\}.$$

The operators B_ν, D_ν do not enter because the controller state is zero. If $G_\nu \le \nu^{-1/6}$, take $c = \nu^{2/3}$; the bound is $C\nu^{1/3}$. Otherwise take $c = (\nu/G_\nu)^{4/7}$. The collision and feedback terms both equal $\nu^{3/7}G_\nu^{4/7}$, and the transport term is no larger in this regime. Hence the infimum of the displayed residual is bounded by the right-hand side of (7.16) without M_*. Exponential stability gives $\|\mathbb{A}_\nu^{-1}\| \le M_*/g_\nu$; apply that inverse to the residual. A fixed positive lower bound for g_ν, with the $\nu^{1/3}$ term tending to zero, forces the stated gain lower bound. □

A large actuator norm $\|B_\nu\|$ alone does not evade this conclusion while the semigroup prefactor remains uniformly bounded in the same state norm. Rescaling the controller norm with ν changes the hypothesis and cannot be used to claim the same uniform estimate. The exponent $3/4$ comes from the velocity-volume estimate for the field together with the cost of velocity diffusion; it is not asserted to be sufficient or optimal among all observation mechanisms.

7.3 An exact obstruction for velocity-even gyroaverages

For the usual electrostatic Bessel gyroaverage, J_s is even in parallel velocity. In that case the gain condition above can be strengthened: one can cancel the field exactly before estimating the residual.

Theorem 7.6 (Failure of uniform relaxation under bounded observations)**.** *Assume that one species has $J_s(\theta,-v,\mu) = J_s(\theta,v,\mu)$ and satisfies* (7.3) *on both velocity branches near zero energy. Use either collision operator admitted by Theorem 7.3. Fix an integer $m \geq 0$, and let*

$$F_\nu : \mathcal{H} \to \mathbb{C}^m, \qquad \mathcal{O}_\nu h = (Qh, F_\nu h)$$

be bounded for each ν; their norms need not be uniform. Replace Q in both observation slots of (7.14) *by $\mathcal{O}_\nu$, with arbitrary bounded controller operators for each ν. Suppose that the resulting realization contains the interior packet vectors and that* (7.15) *holds in fixed state norms. Then*

$$g_\nu \leq C_m M_* \nu^{1/3}, \tag{7.17}$$

independently of every sensor and feedback gain. In particular, no uniform positive decay rate with a uniform prefactor is possible as $\nu \downarrow 0$, even if these gains diverge.

Proof. Choose a fixed $(m+1)$-dimensional space of smooth amplitudes on $W' \Subset W$. On the positive branch use (3.8) with its phase factor removed. Reflect the function to the negative branch and change its sign there; divide by $\sqrt{2}$ to preserve the amplitude norm. The resulting subspace $\mathcal{A}_c$ has dimension $m+1$, is odd in v, and consists of smooth interior functions supported at $E \leq Kc$. The field measure and J_s are even in v, hence

$$Qu = 0 \qquad (u \in \mathcal{A}_c).$$

All invariant amplitude factors are constant along each characteristic. Differentiating the spatial envelope therefore costs $O(\sqrt{c})$; the uncancelled drift costs $O(c)$ by (7.3). The unphased velocity Hessian costs $O(c^{-1})$. The proofs of Lemma 7.1 and Proposition 7.2, without the frequency phase, consequently give

$$\|L_\nu u\| \leq C_m(\sqrt{c} + \nu/c)\|u\|, \qquad u \in \mathcal{A}_c.$$

This estimate is uniform on the subspace. Since $\dim \mathcal{A}_c = m+1$, its intersection with $\ker F_\nu$ contains a nonzero vector. Normalize that vector to obtain $u_{c,\nu}$ with $\mathcal{O}_\nu u_{c,\nu} = 0$. For the full controller vector $(u_{c,\nu}, 0)$, every feedback and observation term is exactly zero, so its residual at zero is bounded by the preceding expression. Choose $c = \nu^{2/3}$ and use $\|\mathbb{A}_\nu^{-1}\| \leq M_*/g_\nu$. This proves (7.17). No estimate on a sensor or controller norm enters the argument. □

The number of extra scalar measurements is fixed in this theorem; it may not increase as $\nu \to 0$. The sensor functionals themselves may vary with ν. Measurements of an entire odd velocity distribution, unbounded observation, or a semigroup prefactor that diverges with ν fall outside the conclusion. The general gain estimate in Theorem 7.5 remains relevant without the velocity symmetry, whereas (7.17) is stronger for the displayed electrostatic Bessel models.

8 Approximation, observation, and feedback

8.1 Approximate evolution

For $u \in \mathcal{D}(L)$ and $t \geq 0$, variation of constants gives

$$S(t)u - e^{zt}u = \int_0^t e^{zs} S(t-s)(L-z)u \, \mathrm{d}s. \tag{8.1}$$

The integral is bounded uniformly for t in a fixed finite interval when z is fixed. Thus the subspaces above approximate scalar evolution, uniformly over their unit spheres. This does not rely on a spectral mapping theorem.

For a bounded operator A, let $\|A\|_{\mathrm{ess}} = \inf_{K \text{ compact}} \|A - K\|$.

Theorem 8.1 (Semigroup spectra and essential norms)**.** *For either generating realization, with arbitrary bounded drive, $\|S(t)\|_{\mathrm{ess}} \geq 1$ for every $t \geq 0$. This lower bound holds in any equivalent Hilbert norm. If $G = 0$, then*

$$\|S(t)\|_{\mathrm{ess},M} = \|S(t)\|_M = 1. \tag{8.2}$$

Moreover, for $t > 0$,

$$\sigma(S(t)) = \sigma_{\mathrm{sap}}(S(t)) = \{\zeta : |\zeta| \leq 1\} \qquad \textit{for absorption}, \tag{8.3}$$

$$\sigma(S(t)) = \sigma_{\mathrm{sap}}(S(t)) = \{\zeta : |\zeta| = 1\} \qquad \textit{for periodic boundaries}, \tag{8.4}$$

when $G = 0$. With drive, the respective sets remain contained in $\sigma_{\mathrm{sap}}(S(t))$.

Proof. For a singular sequence at $\mathrm{i}\omega$, (8.1) yields $S(t)u_n - e^{\mathrm{i}\omega t}u_n \to 0$. Every compact K has $Ku_n \to 0$, so $\|S(t) - K\| \geq 1$. Equivalent norms preserve weak convergence and vanishing residuals; normalize the sequence in the new norm to obtain the same lower bound. Contractivity or unitarity gives (8.2).

The same argument at $\Re z < 0$ puts every nonzero point of the open unit disk into $\sigma_{\mathrm{sap}}(S(t))$ for absorption; the axis supplies its boundary. To include zero, choose $z_j \to -\infty$ and then a sufficiently low-energy packet for each j so that both the residual in (8.1) and $|e^{tz_j}|$ tend to zero. The shrinking energy supports ensure weak nullity. A contraction has spectrum in the closed disk. A unitary operator has spectrum on the circle, and the axis packets supply every point of that circle. □

8.2 Finite-dimensional approximation

Regard the complex Hilbert space as a real normed space and let $B_M = \{h : \|h\|_M \leq 1\}$. An encoder may be any continuous map $\mathcal{E} : B_M \to \mathbb{R}^n$; a decoder $\mathcal{D} : \mathbb{R}^n \to \mathcal{H}$ need not be linear or continuous. This includes encoding an initial distribution, evolving a finite-dimensional latent state for a fixed time, and decoding it.

Theorem 8.2 (Worst-case approximation error)**.** *For each finite n and fixed $t \geq 0$,*

$$\inf_{\substack{\mathcal{E}: B_M \to \mathbb{R}^n \text{ continuous} \\ \mathcal{D}: \mathbb{R}^n \to \mathcal{H}}} \sup_{h \in B_M} \|S(t)h - \mathcal{D}(\mathcal{E}(h))\|_M \geq 1. \tag{8.5}$$

If $G = 0$, equality holds.

Proof. Fix $\varepsilon > 0$. Theorem 3.1, norm equivalence, and (8.1) give an infinite-dimensional subspace on which $\|S(t)u\|_M \geq (1-\varepsilon)\|u\|_M$. Restrict to the unit sphere of any real $(n+1)$-dimensional subspace. By the Borsuk–Ulam theorem, the encoder takes the same value at an antipodal pair $u, -u$. If y is their common decoded value, then

$$\max\{\|S(t)u - y\|_M, \| - S(t)u - y\|_M\} \geq \|S(t)u\|_M \geq 1 - \varepsilon.$$

Let $\varepsilon \downarrow 0$. This is the standard antipodal argument for nonlinear widths [15, Proposition 1], applied to the uniformly persistent kinetic subspaces. When $G = 0$, the zero decoder has error at most one by contractivity. □

The bound concerns the full unit ball; a compact or regularity-bounded class may admit finite-dimensional approximation. It is not a lower bound for the electrostatic field alone.

8.3 Unobserved states and bounded feedback

Let $C_0 : \mathcal{H} \to Y_0$ be compact and define $\mathcal{O}h = (Qh, C_0 h)$ in $Y = \mathcal{X} \oplus Y_0$. Every bounded finite-dimensional measurement can be included in C_0.

Theorem 8.3 (Observation and stabilization)**.** *For every $T > 0$,*

$$\inf_{\|h\|_M = 1} \int_0^T \|\mathcal{O}S(t)h\|_Y^2 \, \mathrm{d}t = 0. \tag{8.6}$$

More generally, let A_c generate a strongly continuous semigroup on a Hilbert space Z, and let $F_0 : Y \to \mathcal{H}$, $F_1 : Z \to \mathcal{H}$, $F_2 : Y \to Z$ be bounded. The closed-loop generator

$$\mathscr{A}\begin{pmatrix} h \\ y \end{pmatrix} = \begin{pmatrix} Lh + F_0\mathcal{O}h + F_1 y \\ A_c y + F_2 \mathcal{O}h \end{pmatrix}, \qquad \mathcal{D}(\mathscr{A}) = \mathcal{D}(L) \oplus \mathcal{D}(A_c), \tag{8.7}$$

has $\mathrm{i}\mathbb{R} \subset \sigma_{\mathrm{sap}}(\mathscr{A})$. It therefore cannot generate a uniformly exponentially stable semigroup in any equivalent Hilbert norm.

Proof. Choose a normalized low-energy sequence at $\mathrm{i}\omega$. The field estimate and compactness give $\mathcal{O}u_n \to 0$. Equation (8.1) makes $\mathcal{O}S(t)u_n - e^{\mathrm{i}\omega t}\mathcal{O}u_n \to 0$ uniformly on $[0, T]$, proving (8.6). The block operator is a bounded perturbation of $L \oplus A_c$, so it is a generator. For $(u_n, 0)$ its residual at $\mathrm{i}\omega$ is

$$\begin{pmatrix} (L - \mathrm{i}\omega)u_n + F_0\mathcal{O}u_n \\ F_2\mathcal{O}u_n \end{pmatrix} \longrightarrow 0.$$

The vectors are weakly null. Exponential stability would put the entire imaginary axis in the resolvent, a contradiction. □

The controller space may be infinite-dimensional. The obstruction comes from observing the state only through $\mathcal{O}$. The theorem does not cover unbounded boundary actuation, velocity derivatives, or observations that remain large on the localized states.

9 Consequences for kinetic reduction

The spectral sets distinguish boundary loss from the kinetic continuum. Periodic evolution preserves the quadratic energy, whereas absorbing evolution loses it through outflow. Even so, arbitrarily slow passing particles keep the absorbing semigroup at distance one from every compact operator at every finite time, so absorption does not give uniform relaxation.

The low-energy states have order-one kinetic norm but vanishing field response. A closure may therefore approximate the field accurately while missing the full distribution. Theorems 8.2 and 8.3 limit uniform claims in the energy norm, not approximation on smoother data or in weaker observables.

Theorem 6.5 locates asymptotically passing approximate eigenvectors in phase space. Under its stated coupling condition, the field cannot sustain such a sequence inside a fixed positive excess-energy region. The proof uses compactness after a bounded flight time and does not require compact quasineutrality or a constant parallel metric.

Velocity diffusion does not remove the obstruction at a rate uniform in the collision frequency for either operator considered here. Theorem 7.5 gives a necessary gain scale, and Theorem 7.6 rules out uniform relaxation for every gain when an even gyroaverage permits exact field cancellation. Matching decay bounds and optimality of the frequency-dependent exponent remain open, as do extensions to electromagnetic constraints and unbounded observation or actuation.

10 Checks of the packet estimates

The accompanying scripts evaluate the estimates rather than time evolution. For $B = \beta = m = 1$ and zero drift, Maxwellian quadrature gives a field exponent of 0.7499994, consistent with (3.2). With fixed smooth amplitudes and the Ornstein–Uhlenbeck operator (7.1), the fitted derivative-cost exponents on $10^{-7} \le c \le 10^{-2}$ are -1.0000000 at $\omega = 0$ and -1.99924 at $\omega = 1$. Increasing the Gauss–Legendre orders from 48 to 64 changes the computed norms by at most 4.7×10^{-11} relatively. The script also checks the two choices of c used in Theorem 7.5. These calculations support the displayed packet estimates and residual bound; the Landau extension itself rests on the analytical kernel and derivative estimates. They do not compute collisional spectra or simulate feedback.

A A circular coefficient family

An explicit family illustrates how the hypotheses can be checked without identifying the coefficients with a particular simulation code. Let

$$B(\theta) = \frac{b_0}{1 + \epsilon \cos\theta}, \qquad \beta(\theta) = \beta_0, \qquad 0 < \epsilon < 1, \tag{A.1}$$

where $b_0, \beta_0 > 0$. Its minimum and maximum are $b_0/(1+\epsilon)$ and $b_0/(1-\epsilon)$. More generally, the arguments allow any positive smooth periodic β.

For a nonzonal field-line block one may prescribe

$$k_\perp^2 = (k_x + \Lambda(\theta) k_y)^2 + g(\theta) k_y^2, \qquad \Lambda(\theta) = \widehat{s}\theta + p(\theta), \tag{A.2}$$

with p, g smooth periodic and $g \ge g_- > 0$. On every bounded interval these are smooth coefficients.

The Bessel multiplier

$$J_s = J_0\left(\frac{k_\perp\sqrt{2m_s\mu/B}}{|q_s|}\right)$$

has modulus at most one. It is smooth even where $k_\perp = 0$, since J_0 is an entire function of the square of its argument. Its Maxwellian square average is

$$\Gamma_s = e^{-b_s} I_0(b_s), \qquad b_s = \frac{m_s k_\perp^2}{q_s^2 B^2}.$$

A prescribed field operator still needs its own polarization lower bound. For example $C = c\mathrm{Id}$ with $c > \sum_s q_s^2$ gives $M \geq (1 - \sum_s q_s^2/c)\mathrm{Id}$. Other constraints, including those approaching a singular wavenumber, must not be assigned this bound automatically.

Drifts $\Omega_s = E_s a_s(\theta) + \mu b_s^{\mathrm{d}}(\theta)$ with smooth real coefficients satisfy the derivative hypothesis (7.3); affine energy drives satisfy (2.6). The collisionless conclusions consequently apply on an absorbing interval. On a periodic cell the complete coefficients must be periodic. In particular, (A.2) with $\widehat{s}k_y \neq 0$ is not a periodic single-block coefficient and cannot be treated as one merely because B is periodic. The choice $k_y = 0$ removes this issue. The distinction between a periodic block and a linked field-line boundary is also important in computational formulations [12].

For (A.1), a trapped pitch obeys $(1-\epsilon)/b_0 < \lambda < (1+\epsilon)/b_0$. Set

$$k^2 = \frac{1 + \epsilon - \lambda b_0}{2\epsilon} \in (0, 1).$$

The change of variables $\sin(\theta/2) = k \sin u$ in (4.1) gives

$$\mathcal{T}_s(\lambda) = \frac{4\sqrt{m_s}}{\beta_0\sqrt{\epsilon}} \int_0^{\pi/2} \frac{\sqrt{1 + \epsilon - 2\epsilon k^2 \sin^2 u}}{\sqrt{1 - k^2 \sin^2 u}}\, \mathrm{d}u. \tag{A.3}$$

Writing $K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 u)^{-1/2}\, \mathrm{d}u$, one obtains the explicit bounds

$$\frac{4\sqrt{m_s(1-\epsilon)}}{\beta_0\sqrt{\epsilon}} K(k) \leq \mathcal{T}_s(\lambda) \leq \frac{4\sqrt{m_s(1+\epsilon)}}{\beta_0\sqrt{\epsilon}} K(k). \tag{A.4}$$

At a compact pitch interval these give finite positive bounds. As $k \uparrow 1$, splitting the integral near $u = \pi/2$ and comparing with $\int_0^1 ((1-k^2) + x^2)^{-1/2}\, \mathrm{d}x$ proves logarithmic divergence. Thus the same family supplies regular trapped orbits, a separatrix, and the two distinct passing limits used in the paper. No benchmark growth rate, transport coefficient, or production-code identity follows from this coefficient example.

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