Repetition graphs and the growth of pattern-avoiding sequences

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Abstract

We study ascent sequences avoiding patterns of the form $012\cdots(d+u)d$. For every fixed nonempty family of these patterns, each using at least three distinct values, the number $A(n)$ of avoiding sequences satisfies $\log A(n) = n\log n - 2n\log\log n + O(n)$. The proof associates a graph with consecutive occurrences of equal values. A bound on its chromatic number allows us to reconstruct a sequence from one occurrence of each value and a controlled amount of additional data. Permutation bounds give the upper estimate; appending distinct larger entries to suitable prefixes gives the lower estimate. The ordinary and exponential generating functions are not D-finite, and the latter is entire of infinite order. Families containing a pattern with fewer than three values instead have explicit rational ordinary generating functions. We also determine the coefficient of the linear term in $\log A(n)$ for simultaneous avoidance of 0120, 0121, and 0122. For weak ascent sequences and inversion sequences, we obtain the corresponding linear terms and concentration estimates for the number of repetitions.

Keywords. Ascent sequences; pattern avoidance; logarithmic asymptotics; D-finite series; inversion sequences.

1 Introduction

An ascent sequence is a nonnegative integer word $x = x_1\cdots x_n$ with $x_1 = 0$ and

$$x_i \le 1 + \operatorname{asc}(x_1\cdots x_{i-1}) \qquad (2 \le i \le n), \tag{1.1}$$

where $\operatorname{asc}(x) = |\{i : x_i < x_{i+1}\}|$. These words encode $(2+2)$-free posets and are counted by the Fishburn numbers [1]. Their pattern avoidance was first studied systematically by Duncan and Steingrímsson [2].

Reduction replaces the distinct values of a word, in increasing order, by $0, 1, \ldots$. A pattern is a nonempty reduced word; it occurs when a subsequence has that reduction. For a finite or infinite family $\mathcal{P}$, let $A_{\mathcal{P}}(n)$ count its avoiding ascent sequences, including the empty word. All families are fixed as $n \to \infty$, all logarithms are natural, and $\log 0 = -\infty$ in upper bounds.

For $B \subseteq \mathbb{N}^2$, where $\mathbb{N} = \{0, 1, \ldots\}$, write $A_B(n)$ for the number of ascent sequences avoiding every pattern

$$R_{d,u} = 0\,1\,\cdots\,(d+u)\,d, \qquad (d,u) \in B.$$

Thus $R_{d,u}$ consists of an increasing sequence of $d+u+1$ values followed by a second occurrence of d. The counts $W_B(n)$ and $I_B(n)$ refer to weak ascent sequences and inversion sequences. A weak ascent sequence satisfies (1.1) with asc replaced by $\operatorname{wasc}(x) = |\{i : x_i \le x_{i+1}\}|$; an inversion sequence satisfies $0 \le x_i \le i-1$. We reserve the unqualified term ascent sequence for (1.1).

Theorem 1.1. *Let $B \subseteq \mathbb{N}^2$ be nonempty.*

(i) If every $(d,u) \in B$ satisfies $d+u \geq 2$, then

$$\log A_B(n) = n\log n - 2n\log\log n + O_B(n). \tag{1.2}$$

Otherwise the ordinary generating function for $A_B(n)$ is one of the rational functions in Theorem 6.3.

(ii) If $(0,0) \notin B$, then

$$\log W_B(n) = n\log n - 2n\log\log n + (\log 2 - 1)n + o(n), \tag{1.3}$$

$$\log I_B(n) = n\log n - n\log\log n - n + o(n). \tag{1.4}$$

If $(0,0) \in B$, each of the three kinds of sequence has exactly one avoider at every length.

(iii) Every counting sequence in (1.2)–(1.4) is non-P-recursive. Its ordinary generating function has radius zero; its exponential generating function is entire of infinite order; and neither is D-finite.

Liu, Kitaev and Zhang [6] leave the enumeration of 0120-avoiding sequences, 0121-avoiding sequences, and sequences avoiding both patterns open. Theorem 1.1 determines their logarithmic asymptotics and rules out linear recurrences of fixed finite order with polynomial coefficients. Appendix A gives a recurrence whose state space grows with the sequence length. The theorem also covers 0122, whose succession rules appear in Section 3 of Mansour's paper on 0132-avoidance [7].

For the upper bound, we join consecutive occurrences of each value by arcs and define a graph on these arcs in Section 4. Avoidance of any one $R_{d,u}$ bounds its clique number after separating a fixed number of color classes. A geometric representation then allows us to apply the coloring theorem of Davies, Krawczyk, McCarty and Walczak [5]. Reading the colored arcs backwards reconstructs the word from the last occurrence of each value and two arrays of colors. If the word has k distinct values and $\delta = n-k$, each array has exactly δ nonzero entries. The retained entries reduce to a permutation $\pi \in S_k$ satisfying

$$\pi_j \leq j + \delta, \qquad \operatorname{des}(\pi) \leq \delta + 1. \tag{1.5}$$

Counting these permutations gives the upper bound. For the lower bound, we choose an avoiding prefix w such that appending distinct entries larger than $\max(w)$ preserves avoidance. The quantity $\Delta(w) = 1 + \operatorname{asc}(w) - \max(w)$ determines which orders of these new entries satisfy the ascent condition. The following theorem separates the two requirements: reconstruction with at most exponentially many auxiliary records, and prefixes for which $\Delta(w)$ grows in proportion to their length.

Theorem 1.2 (Extraction and extension)**.** *Suppose the ascent sequences avoiding $\mathcal{P}$ admit an injective encoding as in Definition 2.3. Suppose also that, for every sufficiently large integer m, there is such a sequence w_m for which appending any distinct entries larger than $\max(w_m)$ preserves avoidance, and*

$$|w_m| \leq C_0 m, \qquad \Delta(w_m) \geq c_0 m \tag{1.6}$$

for fixed $C_0, c_0 > 0$. Then

$$\log A_{\mathcal{P}}(n) = n\log n - 2n\log\log n + O(n). \tag{1.7}$$

Both generating functions are non-D-finite, and the exponential generating function is entire of infinite order.

Equivalently, for some $c, C > 0$ and all large n,

$$\frac{c^n n!}{(\log n)^{2n}} \le A_{\mathcal{P}}(n) \le \frac{C^n n!}{(\log n)^{2n}}. \tag{1.8}$$

The theorem also applies to avoidance of 100, of 110, and of $\{000, 100, 110\}$. For the first two classes, Conway, Conway, Elvey Price and Guttmann estimated growth of the form $C(\alpha n)!\mu^n n^g$ with $\alpha \approx 3/4$ [3, Sections 6–7]. In contrast, (1.7) gives $\log A(n)/(n \log n) \to 1$. Our lower bound uses their prefix consisting of two copies of $01 \cdots (m-1)$, but permits a longer continuation; the reconstruction argument supplies a matching upper bound at the scale of (1.7). This changes the asymptotic conclusion, not their computed coefficients.

2 Permutation extraction and the upper bound

Write $\mathrm{dist}(x)$ for the number of distinct values and $\mathrm{des}(x) = |\{i : x_i > x_{i+1}\}|$. Retain one occurrence of each value and replace the retained values by their ranks in $\{1, \dots, k\}$. The resulting permutation records their relative order. The retained subsequence need not itself be an ascent sequence.

Lemma 2.1. *Let x be an ascent sequence of length n, with k distinct values, and let $\delta = n - k$. Every permutation π obtained by retaining one occurrence of each value satisfies* (1.5).

Proof. If the retained positions are $t_1 < \cdots < t_k$, then $t_j \le j + \delta$. Since $x_i \le i - 1$ and the rank of a nonnegative value is at most that value plus one,

$$\pi_j \le x_{t_j} + 1 \le t_j \le j + \delta.$$

Put $a = \mathrm{asc}(x)$ and $M = \max(x)$. The ascent condition gives $M \le a + 1$, while the presence of k distinct values gives $M \ge k - 1$. Hence $a \ge k - 2$ and $\mathrm{des}(x) \le n - 1 - a \le \delta + 1$. Each descent between consecutive retained entries spans an adjacent descent in x; these intervals have disjoint sets of adjacent pairs. Thus deletion cannot increase the number of strict descents, proving the second bound. □

For integers $k \ge 1$ and $b, d \ge 0$, let $P(k; b, d)$ count the permutations of $\{1, \dots, k\}$ satisfying

$$\pi_j \le j + b \quad (1 \le j \le k), \qquad \mathrm{des}(\pi) \le d.$$

Lemma 2.2 (Displacement and descents)**.** *For all $k \ge 1$ and $b, d \ge 0$,*

$$P(k; b, d) \le (2\mathrm{e})^k \left(1 + \frac{b(d+1)}{k}\right)^k. \tag{2.1}$$

Proof. Cut at the descents into increasing runs of lengths $\ell_1, \dots, \ell_q$, where $q \le d + 1$ and $\sum_i \ell_i = k$. A run of length ℓ following a positions uses values at most $a + \ell + b$. All a earlier values lie in the same interval, leaving at most $\ell + b$ unused values. Choosing their set determines the run, so a fixed composition contributes at most $\prod_i \binom{\ell_i + b}{\ell_i}$. This remains an upper bound when the interval extends beyond k.

The bound $\binom{\ell+b}{\ell} \le [\mathrm{e}(1 + b/\ell)]^\ell$ and the concavity of $f_b(t) = t \log(1 + b/t)$, with $f_b''(t) = -b^2/[t(t+b)^2] \le 0$, give

$$\prod_i \binom{\ell_i + b}{\ell_i} \le \mathrm{e}^k \exp\left(\sum_i \ell_i \log(1 + b/\ell_i)\right) \le \mathrm{e}^k (1 + bq/k)^k.$$

Summing over at most 2^{k-1} compositions and using $q \le d + 1$ proves the claim, also for $b = 0$. □

Definition 2.3. For a class $\mathcal{C}$ of words, consider the following encoding. For each $x \in \mathcal{C}_n$, choose one occurrence of each value by a fixed rule, let $\pi(x)$ be their relative order, and assign auxiliary data $\eta(x)$ in a set E_n. Require $|E_n| \le K^n$ for a fixed constant K and require the map

$$x \longmapsto (S(x), \pi(x), \eta(x)) \tag{2.2}$$

to be injective. Here $S(x)$ is the actual set of values of x, so the encoded data determine the word itself, not just its reduction.

The actual value set need not be consecutive. For ascent sequences it lies in $\{0, \dots, n-1\}$, giving $\binom{n}{k}$ possibilities.

Proposition 2.4. *Suppose a class $\mathcal{C}$ of ascent sequences admits the encoding in Definition 2.3 with constant K. Let $C_{n,k}$ count its words with exactly k distinct values. Then*

$$C_{n,k} \le K^n \binom{n}{k} (2\mathrm{e})^k \left(1 + \frac{(n-k)(n-k+2)}{k}\right)^k. \tag{2.3}$$

Proof. Choose the value set, then apply Lemmas 2.1 and 2.2 with $b = n-k$ and $d = n-k+1$. Each pair admits at most K^n auxiliary records. □

Lemma 2.5 (Uniform optimization)**.** *As $n \to \infty$,*

$$\max_{0 \le \delta \le n-1} (n-\delta) \log\left(1 + \frac{\delta(\delta+2)}{n-\delta}\right) \le n \log n - 2n \log\log n + O(n). \tag{2.4}$$

The constant in $O(n)$ is absolute.

Proof. Denote the maximized expression by $F_n(\delta)$. For $\delta < \sqrt{n}$, $\log(1+y) \le y$ gives $F_n(\delta) = O(n)$ uniformly. Otherwise $n - \delta + \delta(\delta+2) \le 4\delta^2$. With $t = \delta/n$ and $L = \log n$,

$$\begin{aligned} F_n(\delta)/n &\le L - g_L(t) - (1-t)\log(1-t) + (1-t)\log 4, \\ g_L(t) &= tL + 2(1-t)\log(1/t). \end{aligned} \tag{2.5}$$

The last two terms are uniformly bounded. Since

$$g_L'(t) = L + 2\log t + 2 - 2/t, \qquad g_L''(t) = 2/t + 2/t^2 > 0,$$

and g_L' tends to $-\infty$ at zero and is positive at one, there is a unique minimizer $t_L \in (0,1)$. Setting $y_L = 2/t_L$ gives $y_L + 2\log y_L = L + 2 + 2\log 2$, hence first $L/2 < y_L < 2L$ and then $y_L = L + O(\log L)$. The stationary equation yields

$$g_L(t_L) = 2 - 2t_L + 2\log(1/t_L) = 2\log L + 2 - 2\log 2 + O(\log L/L).$$

This proves (2.4). To retain the repetition scale, set $u = \delta L/n$ and use $g_L(t) = 2\log L + u - 2\log u - 2t\log(1/t)$. The bounds $t\log(1/t) \le 1/\mathrm{e}$ and $-(1-t)\log(1-t) \le 1/\mathrm{e}$ give

$$F_n(\delta) \le nL - 2n\log L + n(2\log u - u + C_*), \qquad C_* = \log 4 + 3/\mathrm{e}, \tag{2.6}$$

for $\sqrt{n} \le \delta < n$. □

Theorem 2.6 (Extraction upper bound)**.** *If a class $\mathcal{C}$ of ascent sequences admits the encoding in Definition 2.3, then*

$$|\mathcal{C}_n| \le \exp(n\log n - 2n\log\log n + O(n)). \tag{2.7}$$

Proof. The factors K^n, $\binom{n}{k}$ and $(2\mathrm{e})^k$ in (2.3) have logarithms $O(n)$ uniformly in k. Apply Lemma 2.5; summing over k adds at most $\log n$. □

2.1 The number of repetitions in a typical word

Corollary 2.7. *Suppose $\mathcal{C}$ is a class of ascent sequences admitting the encoding in Definition 2.3, and*

$$\log|\mathcal{C}_n| \geq n\log n - 2n\log\log n - C_0 n$$

for all sufficiently large n. There are constants $0 < a < b$ and $\gamma > 0$ such that a uniformly chosen $X_n \in \mathcal{C}_n$ satisfies

$$\Pr\left(a\frac{n}{\log n} \leq n - \operatorname{dist}(X_n) \leq b\frac{n}{\log n}\right) \geq 1 - \exp(-\gamma n) \tag{2.8}$$

for all sufficiently large n.

Proof. For $\delta \geq \sqrt{n}$, (2.3) and (2.6) bound the logarithm of the fixed-δ count by $n\log n - 2n\log\log n + n(2\log u - u + C)$, where $u = \delta\log n/n$ and C is fixed. Since $2\log u - u \to -\infty$ at both ends of $(0,\infty)$, choose a, b so that it is below $-C_0 - C - 3$ outside $[a,b]$. Divide by the assumed lower bound and sum over at most n values. For $\delta < \sqrt{n}$, the count is $\exp(O(n))$ by the optimization proof, and is negligible under the same normalization. □

3 Appending distinct entries: the lower bound

The lower bound comes from prefixes that can be followed by many orders of new, larger values. Two conditions must be checked separately: the resulting word must avoid the forbidden patterns and satisfy the ascent condition. For nonempty w, write $M(w) = \max(w)$ and $\Delta(w) = 1 + \operatorname{asc}(w) - M(w)$.

Definition 3.1. We consider words w for which wv avoids $\mathcal{P}$ whenever v has pairwise distinct entries exceeding $M(w)$. This condition includes the empty continuation and imposes no ascent restriction on v. When w is empty, it requires avoidance for every word of distinct nonnegative entries.

A cut i of a pattern $p = p_1 \cdots p_h$ is admissible when the suffix $p_{i+1} \cdots p_h$ has distinct entries, all larger than the entries of the prefix. Empty parts are allowed. Let $i(p)$ be the least admissible cut and define

$$\kappa(p) = \operatorname{red}(p_1 \cdots p_{i(p)}).$$

Let ε denote the empty word and put $\kappa(\varepsilon) = \varepsilon$. For this statement only, the empty pattern is deemed to occur in every word.

Proposition 3.2. *A word satisfies the condition in Definition 3.1 if and only if it avoids $\kappa(p)$ for every $p \in \mathcal{P}$. Moreover, $\kappa(\kappa(p)) = \kappa(p)$. These words form the largest subclass of* $\mathrm{Av}(\mathcal{P})$ *closed under appending distinct entries larger than all existing entries. Here* Av *refers to all nonnegative words, without the ascent restriction.*

Proof. An occurrence of p after appending distinct larger entries selects an admissible prefix of p from the original word. This prefix has length at least $i(p)$, so the original word contains $\kappa(p)$. The same conclusion holds if the occurrence lies wholly in the original word. Conversely, an occurrence of $\kappa(p)$ can be completed by appending distinct larger entries in the order of the removed suffix. This proves the characterization for finite and infinite families.

Removing a nonempty admissible suffix from $\kappa(p)$ would, together with the suffix already removed, give an earlier admissible cut of p. Hence κ is idempotent. Applying the characterization to these fixed patterns proves closure under appending distinct larger entries. Any subclass with this closure property satisfies Definition 3.1 and is therefore contained in the stated class. □

A pattern ending in a repeated letter is fixed by κ.

Lemma 3.3 (The ascent condition after a prefix)**.** *Let w be a nonempty ascent sequence and let $\pi \in S_K$. The word*

$$w\,(M(w)+\pi_1)\cdots(M(w)+\pi_K)$$

is an ascent sequence if and only if

$$\pi_r \le r + \Delta(w) - 1 - \operatorname{des}(\pi_1\cdots\pi_{r-1}) \qquad (1 \le r \le K). \tag{3.1}$$

The descent count of an empty or one-letter word is zero.

Proof. At $r = 1$, the ascent condition is $\pi_1 \le \Delta(w)$. For $r \ge 2$, the transition from w to the appended entries contributes one ascent, and the preceding appended entries contribute $r - 2 - \operatorname{des}(\pi_1\cdots\pi_{r-1})$ ascents. Substituting in (1.1) and subtracting $M(w)$ gives (3.1). □

Lemma 3.4 (Block permutations)**.** *Let K, b, q be positive integers. Partition K consecutive positions into blocks of size b, except for a possible final smaller block. Write their sizes as $b_1, \ldots, b_J$. Give each position block the matching interval of consecutive values. Partition each block in turn into consecutive groups of size q, with a possible smaller final group; write these sizes as $c_{j,1}, \ldots, c_{j,h_j}$. Require the entries of each group to be increasing. This defines exactly*

$$T(K; b, q) = \prod_{j=1}^{J} \frac{b_j!}{\prod_{\ell=1}^{h_j} c_{j,\ell}!} \tag{3.2}$$

permutations. Each satisfies

$$\pi_r \le r + b - 1, \qquad \operatorname{des}(\pi) \le \lfloor K/q \rfloor. \tag{3.3}$$

Moreover,

$$\log T(K; b, q) \ge K \log(b/q) - K - b \log b. \tag{3.4}$$

Proof. Choosing each group's value set and placing it increasingly gives the multinomial product (3.2), a bijection with ordered set partitions of the prescribed sizes. Displacement within a block is at most $b - 1$. Descents occur only between groups in one block, at most $\lceil b_j/q \rceil - 1 \le \lfloor b_j/q \rfloor$ in block j; summing proves (3.3).

Using $t \log t - t \le \log(t!) \le t \log t$ gives

$$\log T \ge \sum_j b_j \log b_j - K - K \log q.$$

All but the last block have size b; the last costs at most $b \log b$ relative to $K \log b$. This proves (3.4), including $b = 1$. □

Theorem 3.5 (Lower bound from prefixes)**.** *Suppose that, for all sufficiently large integers m, there is an ascent sequence w_m satisfying Definition 3.1 for $\mathcal{P}$, with*

$$|w_m| \le C_0 m, \qquad \Delta(w_m) \ge c_0 m, \tag{3.5}$$

where $C_0, c_0 > 0$ are fixed. Then

$$\log A_{\mathcal{P}}(n) \ge n \log n - 2n \log\log n - O(n). \tag{3.6}$$

The assertion holds for every sufficiently large integer n.

Proof. Set $m = \lfloor n/\log n \rfloor$, $L_m = |w_m|$, $K = n - L_m$ and $\Delta = \Delta(w_m)$. Since $\Delta \le |w_m|$,

$$\Delta = \Theta(n/\log n), \qquad L_m = O(n/\log n), \qquad K = n - O(n/\log n).$$

Choose $b = \lfloor \Delta/2 \rfloor$ and $q = \lceil 2K/\Delta \rceil$. For large n these are positive and satisfy $b + \lfloor K/q \rfloor \le \Delta$. Thus every permutation in Lemma 3.4 satisfies (3.1); shifting it above $M(w_m)$ produces a legal avoiding word. The map is injective by its tail.

Now $b = \Theta(n/\log n)$, $q = O(\log n)$, $b \log b = O(n)$ and $\log(b/q) \ge \log n - 2\log\log n - O(1)$. Apply (3.4). Replacing K by n costs $O(n)$ since $n - K = O(n/\log n)$. The constructed length is exactly n, proving the assertion at every sufficiently large integer. □

Theorems 2.6 and 3.5 prove (1.7); Section 7 proves its analytic consequences.

4 Repetition graphs and reconstruction

To reconstruct a word, it is enough to retain its last occurrence of each value and record how earlier occurrences connect to them. We bound the amount of additional data by coloring a graph on these connections. The construction applies to arbitrary words, independently of the ascent condition.

Let $w = w_1 \cdots w_n$ have k distinct values. For each pair of consecutive occurrences of a value v, draw an arc (p, q, v), where $p < q$, $w_p = w_q = v$, and no occurrence of v lies strictly between p and q. There are exactly $\delta = n - k$ arcs. Define the *repetition graph* $G(w)$ on these arcs by joining (p, q, v) to (p', q', v'), with $p < p'$, precisely when

$$p < p' < q \quad \text{and} \quad v < v'. \tag{4.1}$$

The right endpoint q' does not enter this condition. Thus an edge records a later arc starting inside an earlier one at a larger value. We use this graph only to assign colors that permit reconstruction.

We need one established class of geometric intersection graphs. Following Davies, Krawczyk, McCarty and Walczak [5], a grounded L-graph is an intersection graph of shapes consisting of a vertical segment rising from the horizontal axis and a horizontal segment extending rightwards from its top. Two vertices are adjacent exactly when their shapes intersect.

Lemma 4.1. *The graph $G(w)$ is a grounded L-graph.*

Proof. Represent (p, q, v) by the vertical segment from $(p + 1/3, 0)$ to $(p + 1/3, v + 1)$ and the rightward horizontal segment ending at $(q - 1/3, v + 1)$. For $p < p'$, the later vertical meets the earlier horizontal exactly when $p < p' < q$ and $v' \ge v$. Equal-valued arcs have disjoint interval interiors, excluding $v' = v$. No other intersections are possible: distinct heights separate horizontals, and rightward horizontals cannot meet earlier verticals. The shifts separate shared occurrence positions. Thus the intersection graph is (4.1); equal heights may be perturbed locally without changing it. □

We use the following geometric coloring theorem of Davies, Krawczyk, McCarty and Walczak [5, Theorem 1.1].

Theorem 4.2 (Davies, Krawczyk, McCarty and Walczak)**.** *Every grounded L-graph with clique number ω has chromatic number at most $17\omega^4$.*

For a position p, let $\lambda_w(p)$ be the largest length of a strictly increasing subsequence ending at p.

Lemma 4.3 (A bound on the number of colors)**.** *If w avoids $R_{d,u}$, then $G(w)$ has a proper coloring with at most*

$$C_{d,u} = \begin{cases} d, & u = 0, \\ d + 17u^4, & u \geq 1 \end{cases} \tag{4.2}$$

colors. For $(d, u) = (0, 0)$, there are no arcs.

Proof. First color every arc (p, q, v) with $\lambda_w(p) \leq d$ by $\lambda_w(p)$. This is a proper coloring on these arcs: every edge oriented by $p < p'$ has $v < v'$, whence $\lambda_w(p') \geq \lambda_w(p) + 1$.

For $u \geq 1$, a clique of $u + 1$ remaining arcs with starts $p_0 < \cdots < p_u$ has values $v_0 < \cdots < v_u$ and $p_u < q_0$. Since $\lambda_w(p_0) > d$, there are d increasing entries below v_0 before p_0. These entries, followed by those at $p_0, \ldots, p_u, q_0$, form $R_{d,u}$. Hence the subgraph on the remaining arcs has clique number at most u and admits $17u^4$ further colors by Theorem 4.2.

For $u = 0$, any arc with $\lambda_w(p) > d$ would complete $R_{d,0}$ after an increasing subsequence of length $d + 1$. Thus every arc has already been colored. If also $d = 0$, avoidance of 00 leaves no arcs. □

Lemma 4.4 (Reverse reconstruction)**.** *Given any proper coloring of $G(w)$ with colors $1, \ldots, C$, the word is determined by the subsequence of last occurrences and two arrays in $\{0, 1, \ldots, C\}^n$. Each array has exactly δ nonzero entries.*

Proof. Record outgoing and incoming arc colors as c_j^+ and c_j^-, using zero for a missing arc. Scan right to left, keeping one set of values for each color. Each set contains the values of arcs whose right endpoints have been processed but whose left endpoints have not. If $c_j^+ = 0$, read the next entry of the retained subsequence in reverse order; otherwise remove the least value from the set indexed by c_j^+. Then insert the recovered w_j in the set indexed by c_j^- if $c_j^- > 0$.

For an outgoing arc (j, q, w_j), any other arc currently stored, say (a, b, t), has $a < j < b$. If $t < w_j$, (4.1) makes it adjacent to the outgoing arc, so it cannot have the same color. Thus w_j is the least stored value of its color. Arcs of the same value cannot be stored simultaneously because they join consecutive occurrences. Reverse induction therefore proves that the procedure recovers every entry. Removal precedes insertion at positions with both an incoming and an outgoing arc. Each of the δ arcs contributes once to each array. □

Theorem 4.5 (Reconstruction of words avoiding $R_{d,u}$)**.** *Fix $(d, u) \neq (0, 0)$ and put $C = C_{d,u}$. The words avoiding $R_{d,u}$ admit the encoding in Definition 2.3. More precisely, for length n and $k = n - \delta$ distinct values, at most*

$$\binom{n}{\delta}^2 C^{2\delta} \tag{4.3}$$

auxiliary records suffice, in addition to the actual value set and the relative order of the last occurrences.

Proof. Order arcs by their starts and choose the lexicographically least proper coloring from Lemma 4.3. The value set and the order of the last occurrences determine the retained subsequence. Each color array has $\binom{n}{\delta} C^\delta$ possibilities, and Lemma 4.4 proves injectivity. Finally $\binom{n}{\delta}^2 C^{2\delta} \leq [4 \max(1, C)^2]^n$. □

No efficient coloring algorithm is required. For 0120, $C_{0,2} = 272$; its contribution is $o(n)$ in the logarithm when $\delta = \Theta(n / \log n)$.

Example 4.6. For $w = 010120$, the arcs are $(1, 3, 0)$, $(3, 6, 0)$ and $(2, 4, 1)$. Only the first and third arcs are adjacent. Give the zero arcs color 1 and the one arc color 2. Then

$$(c_j^+) = (1, 2, 1, 0, 0, 0), \qquad (c_j^-) = (0, 0, 1, 2, 0, 1).$$

The retained subsequence is 120. The reverse scan reads $0, 2, 1$, then removes $0, 1, 0$ from the indicated sets, recovering the word. The reconstruction works here without any avoidance assumption.

5 Reconstruction for the avoidance classes

5.1 Avoiding 100 and 110

Lemma 5.1. *In a* 100*-avoiding word, the subword consisting of all nonlast occurrences of each value is weakly increasing. In a* 110*-avoiding word, the subword consisting of all nonfirst occurrences of each value is weakly increasing.*

Proof. An inversion $a > b$ among nonlast occurrences has a later b, giving 100. An inversion $a > b$ among nonfirst occurrences has an earlier a, giving 110. □

Proposition 5.2. *The classes of words avoiding* 100 *and avoiding* 110 *admit the encoding in Definition 2.3.*

Proof. For 100 retain last occurrences; for 110 retain first occurrences. Record the retained positions and the multiplicities in increasing value order. There are at most 2^n sets of positions and 2^{n-1} positive compositions. The value set and retained permutation determine the retained entries. Subtracting one from each multiplicity gives the multiset of deleted entries, whose order is determined by Lemma 5.1. The recorded positions then interleave the two subsequences. Thus $K = 4$ suffices. □

The triple $\{000, 100, 110\}$ also appears in the classification of Callan and Mansour [4]. For the lower bound, use

$$w_m = (0\,1\,\cdots\,m-1)(0\,1\,\cdots\,m-1). \tag{5.1}$$

For $m \geq 2$, this word has length $2m$, $2m-2$ ascents, maximum $m-1$, and $\Delta(w_m) = m$. Each value appears twice, with only smaller entries before its first copy and only larger entries after its second. This excludes all three patterns. Appending distinct larger entries adds no equal pair and cannot place a smaller value after, or a larger value before, an existing pair. Conway and coauthors used this prefix for 110 [3, Section 7]; Lemma 3.4 permits a continuation of length $n - O(n/\log n)$.

Corollary 5.3. *For every* $\mathcal{P} \subseteq \{000, 100, 110\}$ *with* $100 \in \mathcal{P}$ *or* $110 \in \mathcal{P}$*,*

$$\log A_{\mathcal{P}}(n) = n \log n - 2n \log\log n + O(n).$$

Proof. The prefix (5.1) satisfies the hypotheses of Theorem 3.5. For the upper bound, use the encoding for either 100 or 110, whichever belongs to $\mathcal{P}$, and apply Theorem 2.6. □

5.2 Families of patterns $R_{d,u}$

When $d+u \geq 2$, the pattern $R_{d,u}$ uses at least three distinct values and is therefore avoided by the binary ascent sequence

$$v_m = (01)^m. \tag{5.2}$$

This sequence has length $2m$, maximum 1, m ascents, and $\Delta(v_m) = m$.

Lemma 5.4. *If every pattern in $\mathcal{P}$ ends in a letter that has appeared earlier in that pattern, then adjoining distinct larger entries to a $\mathcal{P}$-avoiding word preserves avoidance.*

Proof. Any occurrence using the appended suffix ends there. Its last value has no earlier equal entry in the whole word, contradicting the pattern's repeated final letter. □

Corollary 5.5. *Fix $R_{d,u}$ with $d+u \geq 2$. Let $\mathcal{P}$ be a finite or infinite family of patterns, each using at least three values and ending in a repeated letter. If $R_{d,u} \in \mathcal{P}$, then*

$$\log A_{\mathcal{P}}(n) = n \log n - 2n \log\log n + O(n). \tag{5.3}$$

The constants can be chosen independently of the other patterns in $\mathcal{P}$.

Proof. The binary prefix (5.2) and Lemma 5.4 give a lower bound independent of $\mathcal{P}$. Every $\mathcal{P}$-avoider also avoids $R_{d,u}$, so Theorems 4.5 and 2.6 give an upper bound whose constant depends only on (d,u). □

In particular, (5.3) applies to every nonempty family of patterns $R_{d,u}$ with $d+u \geq 2$, including all seven nonempty subsets of $\{0120, 0121, 0122\}$.

6 Containment and the rational cases

Containment among the patterns $R_{d,u}$ is determined by the coordinatewise order on $\mathbb{N}^2$. This reduces even an infinite forbidden family to finitely many minimal patterns.

Proposition 6.1 (Minimal forbidden patterns)**.** *For all nonnegative parameters,*

$$R_{d,u} \text{ contains } R_{d',u'} \quad \Longleftrightarrow \quad d' \leq d \text{ and } u' \leq u. \tag{6.1}$$

Every $B \subseteq \mathbb{N}^2$ has a finite set $\mathcal{B}$ of minimal elements, and $A_B(n) = A_{\mathcal{B}}(n)$ for all n. The ascent-sequence avoidance class determines this minimal basis uniquely.

Proof. An occurrence of $R_{d',u'}$ must use both copies of the only repeated value d, together with d' of the d smaller entries and u' of the u larger entries. This proves (6.1).

Every point of B lies above a minimal point, found in its finite lower rectangle. Distinct minimal points have distinct first coordinates; ordering these increasingly makes the second coordinates strictly decrease. Thus there are finitely many. Containment gives equality of the avoidance classes. Since each $R_{d,u}$ is itself an ascent sequence, membership of these words recovers the upward closure of B and hence the minimal basis. □

For a finite supplied set, deleting dominated points computes $\mathcal{B}$. For an arbitrary infinite set this is an existence statement: a membership oracle cannot in general certify that all minimal elements have been found.

Theorem 6.2. *Let $B \subseteq \mathbb{N}^2$, with minimal basis $\mathcal{B}$. The following conditions are equivalent:*

(i) every $(d,u) \in \mathcal{B}$ satisfies $d+u \geq 2$ (including the empty basis);

(ii) $\Delta(w)$ is unbounded among the nonempty B-avoiders;

(iii) the class contains prefixes satisfying Definition 3.1 and (1.6);

(iv) $\lim_{n\to\infty} \dfrac{\log A_B(n)}{n \log n} = 1.$

When these conditions hold,

$$n\log n - 2n\log\log n - O(n) \leq \log A_B(n) \leq \log(n!), \tag{6.2}$$

with a lower-bound constant independent of B. Otherwise $\Delta(w) = 1$ for every nonempty avoider, and the counting function is one of the rational cases in Theorem 6.3.

Proof. Under (i), the word $w_m = (01)^m$ avoids B, has $\Delta(w_m) = m$ and length $2m$, and satisfies Definition 3.1 by Lemma 5.4. Thus (i)⇒(iii)⇒(ii). Theorem 3.5 gives the lower bound in (6.2), while $0 \leq x_i \leq i-1$ gives $A_B(n) \leq n!$, proving (iv).

If (i) fails, one of 00, 010, 011 is forbidden. A 00-avoider is $012\cdots(n-1)$. A 010-avoider is weakly increasing: before a first descent the used values are consecutive, so that descent would return to an earlier value and complete 010. A 011-avoider has the positive subword $12\cdots k$ with zeros inserted arbitrarily. Indeed, a repeated positive value completes 011 with the initial zero; after j positive entries there are at most j ascents, forcing the next new positive to be $j+1$.

Each description gives $\operatorname{asc}(w) = \max(w)$, hence $\Delta(w) = 1$. The last two forms each have at most 2^{n-1} words. Thus (ii) and (iv) fail, completing the equivalences. The rational formulas follow below. □

Theorem 6.3 (The complementary rational classes)**.** *Let $\mathcal{B}$ be a nonempty minimal set of forbidden parameters containing a point (d,u) with $d+u \leq 1$. Put $F_{\mathcal{B}}(z) = \sum_{n\geq 0} A_{\mathcal{B}}(n) z^n$ and $q = z/(1-z)$. Then $\mathcal{B}$ and its counting function occur in the following table:*

Minimal basis	$A_{\mathcal{B}}(n)$, $n \geq 1$	$F_{\mathcal{B}}(z)$
$\{(0,0)\}$	1	$1/(1-z)$
$\{(0,1)\}$ *or* $\{(1,0)\}$	2^{n-1}	$(1-z)/(1-2z)$
$\{(0,1),(r,0)\}$, $r \geq 1$	$\displaystyle\sum_{j=0}^{r} \binom{n-1}{j}$	$\displaystyle 1 + \sum_{j=1}^{r+1} q^j$
$\{(1,0),(0,r)\}$, $r \geq 1$	$\displaystyle\sum_{j=0}^{r} \binom{n-1}{j}$	$\displaystyle 1 + \sum_{j=1}^{r+1} q^j$

For each r, the paired classes in the last two rows are related by a length-preserving bijection; the rows coincide when $r = 1$.

Proof. The singleton counts follow from the preceding descriptions (also [2]). An antichain containing $(0,1)$ can have at most one further point, necessarily $(r,0)$; the case $(1,0)$ is symmetric, and $(0,0)$ lies below every other point.

A 010-avoider is $0^{a_0}1^{a_1}\cdots k^{a_k}$ with $a_i \geq 1$. It avoids $R_{r,0}$ exactly when $a_i = 1$ for $i \geq r$; smaller values may repeat. The nonempty generating function is

$$\sum_{j=1}^{r-1} q^j + \frac{q^r}{1-z} = \sum_{j=1}^{r+1} q^j.$$

Extracting coefficients gives the binomial sum. For the bijection, send x to y with $y_1 = 0$ and $y_i = 0$ when $x_i = x_{i-1}$, $y_i = x_i$ when $x_i = x_{i-1} + 1$. Its positive subword is $12 \cdots k$; inversely, x_i counts the positive entries of y through position i. Repetition of a value at least r in x corresponds to a zero after the rth positive entry of y, exactly an occurrence of $R_{0,r}$ in an 011-avoider. □

For nonempty bases with $d+u \geq 2$ for every forbidden parameter, Corollary 5.5 strengthens (6.2) to (1.2). Its upper bound does not apply to unrestricted ascent sequences.

7 Analytic consequences

A formal series is D-finite if it satisfies a nonzero homogeneous linear differential equation with polynomial coefficients. Equivalently, its coefficients are P-recursive: they eventually satisfy a nonzero linear recurrence with polynomial coefficients [8, Theorem 1.5]. For an entire function f, put $M_f(r) = \max_{|z|=r} |f(z)|$ and $\rho(f) = \limsup_{r\to\infty} \log\log M_f(r)/\log r$.

Lemma 7.1. *Every entire D-finite function has finite order. More precisely, there are constants $C > 0$ and an integer $q \geq 0$ such that*

$$M_f(r) \leq C \exp(Cr^{q+1}) \tag{7.1}$$

for all sufficiently large r.

Proof. The zero function is immediate. Otherwise choose an equation $\sum_{j=0}^{s} p_j(z) f^{(j)}(z) = 0$ with $p_s \not\equiv 0$ and $s \geq 1$. Outside a disk containing the zeros of p_s, the vector $Y = (f, f', \ldots, f^{(s-1)})$ satisfies $Y' = B(z)Y$ with rational B and $\|B(z)\| \leq C_1|z|^q$ for some integer $q \geq 0$. Radial integration and Gronwall's inequality give

$$\|Y(re^{i\theta})\| \leq \|Y(Re^{i\theta})\| \exp\left(C_1 \int_R^r t^q \, dt\right).$$

Continuity bounds the initial norm uniformly in θ, proving (7.1). □

Theorem 7.2 (Coefficient growth and D-finiteness)**.** *Let $a_n \geq 0$ be eventually positive, and suppose that for a fixed $c > 0$,*

$$\log a_n = n\log n - cn\log\log n + O(n). \tag{7.2}$$

Then $F(z) = \sum_{n\geq 0} a_n z^n$ has radius zero, while

$$E(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$$

is entire of infinite order. Neither series is D-finite, and (a_n) is not P-recursive.

Proof. Set $b_n = a_n/n!$. Stirling's formula gives

$$\log b_n = -cn\log\log n + O(n). \tag{7.3}$$

Hence $b_n^{1/n} \to 0$ and $a_n^{1/n} \to \infty$, proving the radii. If E had finite order, then $M_E(r) \leq C\exp(Cr^\rho)$ for some $C, \rho > 0$ and all large r. Cauchy's estimate at $r = (n/(C\rho))^{1/\rho}$ would give

$$\log b_n \leq -(n/\rho)\log n + O(n), \tag{7.4}$$

contradicting (7.3). Lemma 7.1 therefore excludes D-finiteness of E.

If F were D-finite, a recurrence $\sum_{j=0}^{s} P_j(n)a_{n+j} = 0$ with polynomials not all zero would, after division by $n!$, become

$$\sum_{j=0}^{s} P_j(n) \left(\prod_{h=1}^{j} (n+h) \right) b_{n+j} = 0.$$

At least one coefficient remains nonzero, making (b_n) P-recursive and E D-finite, a contradiction. The same argument excludes P-recursiveness of (a_n). □

Corollary 7.3. *The ordinary and exponential generating functions for every class in Corollaries 5.3 and 5.5 are not D-finite. In particular, none of $A_{0120}(n)$, $A_{0121}(n)$, and $A_{0120,0121}(n)$ satisfies a scalar linear recurrence of finite order with polynomial coefficients.*

Proof. Apply Theorem 7.2 with $c = 2$. This proves the analytic part of Theorem 1.2 and the assertion for ascent sequences in Theorem 1.1(iii). □

Proposition 7.4. *Under the hypotheses of Theorem 7.2,*

$$\log\log M_E(r) = \Theta(r^{1/c}), \qquad \lim_{r\to\infty} \frac{\log\log\log M_E(r)}{\log r} = \frac{1}{c}. \tag{7.5}$$

Proof. For fixed $A, B > 0$ and all $n \geq n_0$,

$$[A/(\log n)^c]^n \leq b_n \leq [B/(\log n)^c]^n.$$

Nonnegativity gives $M_E(r) = E(r)$. Taking $n = \lfloor \exp((Ar/4)^{1/c}) \rfloor$ yields $b_n r^n \geq 2^n$ for large r, hence $\log\log E(r) \geq (Ar/4)^{1/c} + O(1)$.

For the upper bound set $N = \lceil \exp((2Br)^{1/c}) \rceil$. Then $b_n r^n \leq 2^{-n}$ for $n \geq N$. For fixed sufficiently large $D \geq 1$, every earlier term is at most $(Dr)^N$, including the finite initial part. Therefore $E(r) \leq D'(N+1)(Dr)^N$ for large r, giving $\log\log E(r) \leq (2Br)^{1/c} + O(\log\log r)$. Taking one more logarithm proves the limit. □

Proposition 7.5. *If, more precisely,*

$$\log a_n = n\log n - cn\log\log n + \beta n + o(n), \qquad c > 0,$$

then

$$\lim_{r\to\infty} \frac{\log\log M_E(r)}{r^{1/c}} = \exp((\beta+1)/c). \tag{7.6}$$

Proof. For fixed $0 < \varepsilon < 1$ and large n, the preceding coefficient bounds hold with $A_\varepsilon = \exp(\beta+1-\varepsilon)$ and $B_\varepsilon = \exp(\beta+1+\varepsilon)$. Choose $n = \lfloor \exp(((1-\varepsilon)A_\varepsilon r)^{1/c}) \rfloor$; the selected term is at least $(1-\varepsilon)^{-n}$, giving limiting lower bound $((1-\varepsilon)A_\varepsilon)^{1/c}$. At $N = \lceil \exp(((1+\varepsilon)B_\varepsilon r)^{1/c}) \rceil$, the tail is at most $\sum_{n\geq N}(1+\varepsilon)^{-n}$. The same finite-sum estimate gives limiting upper bound $((1+\varepsilon)B_\varepsilon)^{1/c}$. Let $\varepsilon \downarrow 0$. □

8 The linear term in the logarithm

The preceding upper bounds allow exponentially many auxiliary records. To determine the coefficient of n in the logarithm of the count, we need a sharper estimate: near the relevant number of repetitions, these records must contribute only $o(n)$ to the logarithm.

Definition 8.1. Consider an encoding of the form (2.2) for which, with fixed constants $a \geq 0$ and $K \geq 1$, the auxiliary records for words of length n with $n - \delta$ distinct values number at most

$$\binom{n}{\delta}^{a} K^{\delta}. \tag{8.1}$$

The actual value set is still recorded separately.

The encoding by colored arcs satisfies (8.1) with $a = 2$ and $K = C_{d,u}^{2}$. The encodings for 100 and 110 also satisfy this bound: the retained positions have $\binom{n}{\delta}$ choices, and the multiplicities of the deleted entries have $\binom{n-1}{\delta} \leq \binom{n}{\delta}$ choices. These encodings apply to arbitrary words.

8.1 Comparison with weak ascent sequences avoiding 011

Bényi, Mansour and Ramírez [10, Proposition 2.11 and Theorem 2.12] show that the number of weak ascent sequences of length N avoiding 011 and containing m zeros is

$$B(N, m) = \binom{N - 1 + \binom{m}{2}}{N - m}, \qquad 1 \leq m \leq N. \tag{8.2}$$

Their positive entries are distinct, and $\sum_m B(N, m) = W_{011}(N)$. The uniform estimate below implies the summed logarithmic expansion recorded by Kotesovec in OEIS A098569, with an index shift of one [11].

Lemma 8.2. *Let $\pi \in S_k$ record the relative order of one occurrence of each value in an ascent sequence or a weak ascent sequence of length n, and put $\delta = n - k$. Then*

$$\pi_j \leq j + \delta - \operatorname{des}(\pi_1 \cdots \pi_{j-1}) \qquad (1 \leq j \leq k). \tag{8.3}$$

Consequently $0^{\delta+1}\pi$ is a weak ascent sequence of length $n + 1$ avoiding 011, and there are at most $B(n + 1, \delta + 1)$ possible permutations π.

Proof. Let t_j be the original position of the jth retained entry. For $t \geq 2$, the weak ascent condition gives $x_t + 1 \leq t - \operatorname{des}(x_1 \cdots x_{t-1})$. The ascent condition is stronger because it also subtracts equal adjacent pairs. Deletion cannot increase descents, so

$$\pi_j \leq x_{t_j} + 1 \leq t_j - \operatorname{des}(\pi_1 \cdots \pi_{j-1}) \leq j + \delta - \operatorname{des}(\pi_1 \cdots \pi_{j-1}).$$

If $t_j = 1$, then $\pi_j = 1$. For the prefix $0^{\delta+1}$, the quantity $1 + \operatorname{wasc}(w) - \max(w)$ equals $\delta + 1$. The calculation in Lemma 3.3, with weak ascents in place of ascents, shows that (8.3) is exactly the condition for appending π. Distinct positive entries ensure 011-avoidance, and deleting the initial zeros recovers π. □

Lemma 8.3 (A uniform estimate for $B(N, m)$)**.** *Let $L = \log N$ and $u = mL/N$. Uniformly for $N/L^2 \leq m \leq N/\sqrt{L}$,*

$$\log B(N, m) = NL - 2N \log L + N(2 \log u - u - \log 2 + 1) + o(N). \tag{8.4}$$

The complementary ranges have smaller logarithms than $NL - 2N \log L$ by $\omega(N)$. In particular,

$$\log W_{011}(N) = N \log N - 2N \log \log N + (\log 2 - 1)N + o(N). \tag{8.5}$$

Proof. Put $h = N - m$ and $Q = N - 1 + \binom{m}{2}$. If $m \le N/L^2$, then $Q/h \le CN/L^4$ and $\binom{Q}{h} \le (\mathrm{e}Q/h)^h$ gives $\log B(N,m) \le NL - 4N\log L + O(N)$. If $N/\sqrt{L} \le m < N$, use $Q \le N^2$ and $h(1+\log(N/h)) \le N$ to get $\log B(N,m) \le (N-m)L + N \le NL - N\sqrt{L} + N$. For $m = N$ the term is 1.

In the middle range, uniformly $Q = (m^2/2)(1+o(1))$, $h \asymp N$ and $h^2/Q = O(L^4)$. The binomial product and Stirling's estimate give

$$\log \binom{Q}{h} = h\log(Q/h) + h + O(h^2/Q + \log Q),$$

since $h/Q \to 0$ and $\sum_{j<h} \log(1 - j/Q) = O(h^2/Q)$. Substitute $m = uN/L$. Replacing h by N, except in $-mL$, costs $O(N\log L/\sqrt{L})$, while the product error is $O(L^4)$. Both are uniformly $o(N)$, proving (8.4).

The profile $2\log u - u - \log 2 + 1$ has maximum $\log 2 - 1$ at $u = 2$, attained within $o(1)$ by $m = \lfloor 2N/L \rfloor$. Summing at most N terms adds at most $\log N$ to their largest logarithm, proving (8.5). □

Lemma 8.4 (Moving zeros to the front)**.** *Let $Z(N,m)$ count the weak ascent sequences $0^m v$ of length N whose positive subword v has distinct entries. Then*

$$\frac{B(N,m)}{\binom{N}{m}} \le Z(N,m) \le B(N,m). \tag{8.6}$$

Every word counted by $Z(N,m)$ avoids every pattern $R_{d,u}$ other than 00*.*

Proof. Move all zeros of a weak ascent sequence avoiding 011 to the front, preserving the positive subword v. If v_j originally followed $q \le m$ zeros, the weak ascent condition and the fact that deletion cannot increase descents give

$$v_j \le q + j - 1 - \mathrm{des}(v_1 \cdots v_{j-1}) \le m + j - 1 - \mathrm{des}(v_1 \cdots v_{j-1}),$$

which proves the weak ascent condition after 0^m. Recording the original zero positions recovers each preimage, so every fiber has size at most $\binom{N}{m}$. The other inequality follows by inclusion. A repeated endpoint of $R_{d,u}$ cannot be positive because the positive entries are distinct. It cannot be zero when $d + u > 0$ because no zero follows a positive entry. Only 00 can occur. □

8.2 Upper bounds for the linear term

Theorem 8.5. *A class $\mathcal{C}$ admitting an encoding satisfying* (8.1) *obeys the following bounds, according to the kind of sequence it contains:*

$$\log|\mathcal{C}_n| \le n\log n - 2n\log\log n + (\log 2 - 1)n + o(n) \qquad \textit{ascent or weak ascent,} \tag{8.7}$$

$$\log|\mathcal{C}_n| \le n\log n - n\log\log n - n + o(n) \qquad \textit{inversion sequences.} \tag{8.8}$$

In each case the encoding must be defined on the class being counted.

Proof. Set $L = \log n$ and $\delta = n - k$. Including the value set, the number of records is at most $\binom{n}{\delta}^{a+1} K^\delta$. On any fixed range

$$a_0 n/L \le \delta \le b_0 n/L, \qquad 0 < a_0 < b_0 < \infty, \tag{8.9}$$

its logarithm is at most $(a+1)\delta\log(\mathrm{e}n/\delta) + \delta\log K = O(n\log L/L) = o(n)$ uniformly.

For ascent sequences and weak ascent sequences, Lemmas 8.2 and 8.3 bound the number of retained permutations by $B(n+1, \delta+1)$. With $u = \delta L/n$, they give

$$\log |\mathcal{C}_{n,n-\delta}| \le nL - 2n \log L + n(2 \log u - u - \log 2 + 1) + o(n). \tag{8.10}$$

For inversion sequences, Lemma 9.1 bounds them by $(\delta+1)^k$, giving

$$\log |\mathcal{C}_{n,n-\delta}| \le nL - n \log L + n(\log u - u) + o(n). \tag{8.11}$$

The respective profile maxima are $\log 2 - 1$ and -1.

It remains to show that repetition counts outside (8.9) do not affect the upper bound; no lower bound on $|\mathcal{C}_n|$ is assumed. The number of auxiliary records is at most exponential in n. For ascent sequences and weak ascent sequences, the permutation bounds used in (2.6) give the uniform estimate $2 \log u - u + O(1)$ for the normalized linear term when $\delta \ge \sqrt{n}$; the smaller range contributes $\exp(O(n))$. For inversion sequences, set $N = n+1$, $v = \delta+1$, $t = v/N$ and $u' = v \log N/N$. Then

$$(n-\delta) \log(\delta+1) = N \log N - N \log \log N + N(\log u' - u') + Nt \log(1/t).$$

Here $t \log(1/t) \le 1/\mathrm{e}$, and the records contribute $O(n)$. Both coarse profiles tend to $-\infty$ at either end, so choose a_0 small and b_0 large enough that the outside contribution is below the claimed upper term by n. On (8.9) use the uniform refined profiles. Summing over δ adds $O(\log n)$. □

8.3 Bounds for ascent sequences

Proposition 8.6. *Let B be nonempty, with $s = \min_{(d,u)\in B}(d+u) \ge 2$, and put*

$$H_B(n) = \frac{\log A_B(n)}{n} - \log n + 2 \log \log n.$$

Then

$$\log 2 - 1 + 2 \log\left(\frac{s-1}{s}\right) \le \liminf_{n\to\infty} H_B(n) \le \limsup_{n\to\infty} H_B(n) \le \log 2 - 1. \tag{8.12}$$

For 100, 110, *and* $\{000, 100, 110\}$, *the same bounds hold with $s = 2$.*

Proof. Theorem 8.5 gives the upper bound. For the lower bound put $L = \log n$, $\rho = (s-1)/s$, $q = \lfloor 2n/(sL) \rfloor$ and $w = (0\,1 \cdots s-1)^q$. This prefix has length $\ell = sq$ and $\Delta(w) = m = (s-1)q - s + 2$. It avoids every forbidden pattern because it uses only s values, and Lemma 5.4 shows that appending distinct larger entries preserves avoidance. Set $N = n - \ell + m$; for large n, $1 \le m \le N$.

For every $0^m v$ counted by $Z(N, m)$, add $\max(w)$ to each entry of v and append the result to w. The weak ascent inequality $v_j \le j + m - 1 - \mathrm{des}(v_1 \cdots v_{j-1})$ is precisely the ascent condition after w, by the calculation in Lemma 3.3. The appended entries determine v, so this construction gives distinct avoiding sequences of length n.

Now $m = 2\rho n/L + O(1)$ and $N = n - 2(1-\rho)n/L + O(1)$. Lemmas 8.3 and 8.4 yield

$$\log A_B(n) \ge \log B(N, m) - \log \binom{N}{m} = nL - 2n \log L + n(\log 2 - 1 + 2 \log \rho) + o(n).$$

Here $\log \binom{N}{m} = o(n)$, $N \log N = nL - 2(1-\rho)n + o(n)$ and $m \log N/N \to 2\rho$. For the three other classes, use the prefix (5.1) with $q = \lfloor n/L \rfloor$ in place of m. Its length is $2q$ and its value of Δ is q, so the same calculation applies with $\rho = 1/2$. □

8.4 Simultaneous avoidance of 0120, 0121, and 0122

Let $\mathcal{D}_2 = \{(0,2),(1,1),(2,0)\}$, corresponding to simultaneous $0120, 0121, 0122$-avoidance.

Lemma 8.7. *An ascent sequence avoids $\mathcal{D}_2$ if and only if it is binary, or all entries from the first value at least 2 onwards are distinct and at least 2. Consequently, for $n \geq 1$,*

$$A_{\mathcal{D}_2}(n) = 2^{n-1} + \sum_{\ell=2}^{n-1} \sum_{a=1}^{\lfloor \ell/2 \rfloor} \binom{\ell}{2a} Z(n-\ell+a, a). \tag{8.13}$$

Proof. Before the first entry at least 2, a 0 and a 1 have occurred in that order. A later 0 completes 0120, a later 1 completes 0121, and any repeated value at least 2 completes 0122 with the initial $0, 1$. Conversely, a binary prefix avoids all three patterns, and appending distinct larger entries preserves avoidance by Lemma 5.4.

A binary prefix of length ℓ starting at zero with a ascents has $2a-1$ or $2a$ changes, giving $\binom{\ell-1}{2a-1} + \binom{\ell-1}{2a} = \binom{\ell}{2a}$ choices. For $a \geq 1$ its maximum is 1 and its value of Δ is a. Subtracting 1 from each subsequent entry identifies the permitted continuations with the words counted by $Z(n-\ell+a, a)$. The split at the first value at least 2 is unique. Adding the 2^{n-1} binary words gives (8.13). □

Theorem 8.8. *For ascent sequences avoiding $\mathcal{D}_2$,*

$$\log A_{\mathcal{D}_2}(n) = n \log n - 2n \log\log n - (1 + \log 2)n + o(n). \tag{8.14}$$

Proof. Proposition 8.6 gives the lower bound. In (8.13), use $Z(N,a) \leq B(N,a)$ and put $L = \log n$, $K = n - \ell$, $Q = K + \binom{a+1}{2} - 1$, so $B(n-\ell+a, a) = \binom{Q}{K}$.

If $\ell \leq n/L^2$, then $a \leq \ell/2$, $Q/K \leq Cn/L^4$ and the logarithm of the binomial is at most $nL - 4n \log L + O(n)$. If $\ell \geq n/\sqrt{L}$, use $Q \leq n^2$ and $K(1 + \log(n/K)) \leq n$ to bound it by $(n-\ell)L + n \leq nL - n\sqrt{L} + n$. The factor $\binom{\ell}{2a} \leq 2^\ell$ adds at most $n \log 2$, so both ranges are negligible. In the middle range $n/L^2 \leq \ell \leq n/\sqrt{L}$, the subrange $a \leq n/L^2$ is negligible by the first bound.

On the remaining range, uniformly $Q = (a^2/2)(1 + o(1))$ and $K^2/Q = O(L^4)$. The binomial-product estimate of Lemma 8.3, with $u = aL/n$ and $v = \ell L/n$, gives

$$\log \binom{Q}{K} = nL - 2n \log L + n(2 \log u - v - \log 2 + 1) + o(n).$$

The errors are $O(n \log L/\sqrt{L} + L^4) = o(n)$, and $\log \binom{\ell}{2a} = o(n)$. Since $v \geq 2u$,

$$2 \log u - v - \log 2 + 1 \leq 2 \log u - 2u - \log 2 + 1 \leq -1 - \log 2,$$

with maximum at $u = 1$. Taking the largest of at most n^2 summands adds $2 \log n$; the binary contribution is negligible. This proves the upper bound. □

For fixed a, the number $Z(K+a, a)$ can be computed by a recurrence with finitely many states. Let $T_K(s,q)$ count continuations of length K when s unused positive values are currently allowed, of which q are below the last entry. Then

$$T_0(s,q) = 1, \qquad Z(K+a, a) = T_K(a, 0),$$
$$T_{K+1}(s,q) = \sum_{r=q+1}^{s} T_K(s, r-1) + \sum_{r=1}^{q} T_K(s-1, r-1).$$

Here r is the rank of the next entry among the s allowed unused values. Only reachable states $0 \leq q \leq s$ occur, and $T_{K+1}(0,0) = 0$. Choosing $r > q$ creates an ascent, so the weak ascent bound permits one new largest value and s is unchanged. Choosing $r \leq q$ creates a descent and reduces s by one. In both cases $r-1$ unused allowed values lie below the new last entry. This proves the recurrence used in (8.13).

9 Inversion and weak ascent sequences

9.1 Inversion sequences and an exact linear term

An inversion sequence satisfies $0 \le x_i \le i-1$; its avoidance counts are $I_{\mathcal{P}}(n)$. For recurrences and growth results in length-three avoidance, see Testart [9].

Lemma 9.1. *For $b \ge 0$, at most $(b+1)^k$ permutations $\pi \in S_k$ satisfy $\pi_j \le j+b$. More precisely, for $K \ge m \ge 1$,*

$$|\{\pi \in S_K : \pi_j \le j+m-1\}| = m^{K-m}m!. \tag{9.1}$$

Proof. At position j, the $j-1$ earlier values all lie among the first $\min(k, j+b)$ values, leaving at most $b+1$ choices. For (9.1), there are exactly m choices at each of the first $K-m$ positions; the remaining m values have arbitrary order. □

Theorem 9.2 (Inversion sequences with zero prefixes)**.** *Suppose the inversion sequences avoiding $\mathcal{P}$ admit an encoding satisfying* (8.1)*. Suppose also that appending any distinct positive entries to 0^m preserves avoidance for every $m \ge 1$. Then*

$$\log I_{\mathcal{P}}(n) = n\log n - n\log\log n - n + o(n). \tag{9.2}$$

These hypotheses hold for every nonempty family of patterns $R_{d,u}$ not containing 00*, and for each nonempty subfamily of* $\{100, 110\}$*.*

Proof. Theorem 8.5 gives the upper bound. Set $m = \lfloor n/\log n \rfloor$ and $K' = n-m$. Append to 0^m a permutation of $\{1, \dots, K'\}$ with $\pi_j \le j+m-1$. This inequality is the inversion sequence bound at position $m+j$, and the hypothesis on 0^m ensures avoidance. For large n, $K' \ge m$, and (9.1) gives $m^{K'-m}m!$ words. Stirling's estimate yields

$$\log(m^{K'-m}m!) = K'\log m - m + O(\log m) = n\log n - n\log\log n - n + o(n).$$

Every $R_{d,u}$ except 00 is absent from a constant prefix and ends in a repeated letter, so Lemma 5.4 verifies the condition on appended entries. Any one forbidden $R_{d,u}$ supplies an encoding satisfying (8.1). The same two properties hold for 100 and 110. □

Corollary 9.3 (Number of repetitions in inversion sequences)**.** *Under the hypotheses of Theorem 9.2, for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that a uniformly chosen avoider X_n satisfies*

$$\Pr\left(\left|\frac{(n - \operatorname{dist}(X_n))\log n}{n} - 1\right| > \varepsilon\right) \le \exp(-c_\varepsilon n)$$

for all sufficiently large n.

Proof. Localize to a compact interval as in Theorem 8.5. There (8.11) is uniform, and Theorem 9.2 gives the denominator. The profile $\log u - u$ has unique maximum -1 at $u = 1$, with a fixed gap on $|u-1| \ge \varepsilon$. Summing at most n exponentially smaller terms proves the claim. □

More generally, suppose the inversion sequences avoiding $\mathcal{P}$ admit the encoding in Definition 2.3, and the class contains ascent sequences satisfying Definition 3.1 and (1.6). Then

$$\log I_{\mathcal{P}}(n) = n\log n - n\log\log n + O(n). \tag{9.3}$$

The displacement bound and the optimization for inversion sequences in Theorem 8.5 give the upper bound. For a prefix of length L_m and maximum M_m, we have $b = L_m - M_m \ge$

$\Delta(w_m) = \Omega(m)$. Choose $m = \lfloor n/\log n \rfloor$ and append permutations in matching blocks of size b. Their displacement ensures the inversion sequence bound, and Lemma 3.4, with $q = 1$, gives the required lower bound. The prefixes in (5.1) verify these hypotheses for $\{000, 100, 110\}$. This class forbids long strings of zeros, however, so Theorem 9.2 does not determine its linear term.

9.2 Weak ascent sequences

The same comparison with $B(N, m)$ determines the linear term for weak ascent sequences when all prefixes 0^m permit distinct positive continuations.

Theorem 9.4 (Weak ascent sequences with zero prefixes)**.** *Suppose the weak ascent sequences avoiding $\mathcal{P}$ admit an encoding satisfying* (8.1)*. Suppose also that appending any distinct positive entries to 0^m preserves avoidance for every $m \geq 1$. Then*

$$\log W_{\mathcal{P}}(n) = n \log n - 2n \log\log n + (\log 2 - 1)n + o(n). \tag{9.4}$$

This holds for every nonempty family of patterns $R_{d,u}$ not containing 00*, and for each nonempty subfamily of* $\{100, 110\}$*.*

Proof. Use Theorem 8.5 for the upper bound. With $m = \lfloor 2n/\log n \rfloor$, Lemmas 8.3 and 8.4 give

$$\log Z(n, m) \geq n \log n - 2n \log\log n + (\log 2 - 1)n - o(n),$$

since $\log \binom{n}{m} = O(n \log\log n/\log n) = o(n)$. The hypothesis on 0^m makes each word counted by $Z(n, m)$ an avoider. The specified families satisfy both hypotheses by the argument in Theorem 9.2. □

Corollary 9.5 (Number of repetitions in weak ascent sequences)**.** *Under the hypotheses of Theorem 9.4, for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that a uniformly chosen avoider X_n satisfies*

$$\Pr\left(\left|\frac{(n - \operatorname{dist}(X_n))\log n}{n} - 2\right| > \varepsilon\right) \leq \exp(-c_\varepsilon n)$$

for all sufficiently large n.

Proof. After the compact localization of Theorem 8.5, use the uniform profile (8.10) and the denominator from Theorem 9.4. Its unique maximum at $u = 2$ has a fixed gap outside any ε-neighborhood. Summing over at most n repetition counts gives the bound. □

More generally, suppose the weak ascent sequences avoiding $\mathcal{P}$ admit the encoding in Definition 2.3. If they contain prefixes w_m satisfying Definition 3.1, with length $O(m)$ and $1 + \operatorname{wasc}(w_m) - \max(w_m) = \Omega(m)$, then $\log W_{\mathcal{P}}(n) = n \log n - 2n \log\log n + O(n)$. Indeed, $x_i \leq i - 1$, $\max(x) \leq 1 + \operatorname{wasc}(x)$ and $\operatorname{des}(x) = n - 1 - \operatorname{wasc}(x)$ give the permutation bounds of Lemma 2.1. Ascents and weak ascents agree among the distinct appended entries and at their transition from the prefix. The lower construction therefore applies with $1 + \operatorname{wasc}(w_m) - \max(w_m)$ in place of $\Delta(w_m)$. This proves the estimate for $\{000, 100, 110\}$, but does not determine its linear term.

Forbidding 00 forces $012\cdots(n-1)$ for ascent sequences, weak ascent sequences, and inversion sequences. Every nonempty family of patterns $R_{d,u}$ that does not contain 00 satisfies the weak ascent and inversion sequence estimates above. Theorem 7.2 applies with $c = 2$ and $c = 1$, respectively. Writing E_W and E_I for the corresponding exponential generating functions, Proposition 7.5 gives

$$\log\log M_{E_W}(r) \sim \sqrt{2r}, \qquad \log\log M_{E_I}(r) \sim r.$$

10 Remaining questions

For a nonempty family B with $d + u \geq 2$ for every $(d, u) \in B$, the remaining question for ascent sequences is the existence and value of

$$\beta_B = \lim_{n\to\infty} \left(\frac{\log A_B(n)}{n} - \log n + 2 \log\log n \right).$$

Theorem 8.8 determines this limit for $\mathcal{D}_2$. For 0120, 0121, and their simultaneous avoidance, Proposition 8.6 places all subsequential limits in $[-1 - \log 2, \log 2 - 1]$. Whether these limits exist or agree remains unproved. Near the dominant number of repetitions, the auxiliary records contribute only $o(n)$ to the logarithm. The unresolved part is to count the permutations that can occur as the relative order of retained values in ascent sequences in these classes.

The estimate (1.2) excludes the empty family. For other forbidden patterns, it remains open to characterize the classes admitting an encoding satisfying (8.1). The recurrence below computes coefficients for finite bases of patterns $R_{d,u}$, but its state space grows with the sequence length; it is neither a closed formula nor a P-recurrence.

A A recurrence using increasing subsequences

Fix a nonempty finite basis $\mathcal{B}$, put $D = \max_{(d,u)\in\mathcal{B}} d$ and $U = \max_{(d,u)\in\mathcal{B}} u$, and assign absent minima the value $+\infty$. The recurrence also applies to the rational cases.

For a word w, let $\gamma_q(w)$ be the least terminal value of an increasing subsequence of length q, for $1 \leq q \leq D$, and set $\gamma_0(w) = -1$. For a value i, a lower length $0 \leq q \leq D$ and an upper length $0 \leq \ell \leq U$, consider subsequences of the form

$$v_1 < \cdots < v_q < i < t_1 < \cdots < t_\ell, \tag{A.1}$$

where the displayed inequalities describe the values in their subsequence order. For $\ell \geq 1$, let $m_{i,q,\ell}(w)$ be the least possible terminal value t_ℓ. For $\ell = 0$, set $m_{i,q,0}(w) = i$ if the lower chain followed by i exists, and $+\infty$ otherwise. Values absent from w have only infinite coordinates.

Proposition A.1. *Appending t to a $\mathcal{B}$-avoider creates a forbidden pattern $R_{d,u}$ if and only if $m_{t,d,u}(w) < +\infty$ for some $(d, u) \in \mathcal{B}$. Whether or not a forbidden pattern is created, the arrays for wt satisfy the simultaneous updates*

$$\gamma'_q = \min(\gamma_q,\ t \text{ if } \gamma_{q-1} < t), \qquad 1 \leq q \leq D, \tag{A.2}$$

$$m'_{i,q,0} = \min(m_{i,q,0},\ t \text{ if } i = t \text{ and } \gamma_q < t), \tag{A.3}$$

$$m'_{i,q,\ell} = \min(m_{i,q,\ell},\ t \text{ if } i < t \text{ and } m_{i,q,\ell-1} < t), \qquad 1 \leq \ell \leq U. \tag{A.4}$$

Every right-hand side uses the old arrays; a failed condition contributes no candidate to its minimum. The coordinate γ'_0 remains -1.

Proof. A new occurrence of $R_{d,u}$ must end at the appended position. Removing its final t leaves a subsequence of the form (A.1) with $i = t$, $q = d$, and $\ell = u$, proving the first assertion. A new increasing subsequence extends an old one ending below t, which gives (A.2). A new subsequence counted by $m_{i,q,0}$ ends at $i = t$ after an increasing subsequence of length q, which gives (A.3). When $\ell \geq 1$, the new terminal entry t extends an old subsequence whose terminal value is smaller, giving (A.4). In each case the minimum terminal value suffices to decide whether an extension exists. Old subsequences remain available, and simultaneous updates prevent using the new position twice. □

Let $S(w) = (h, a, \gamma, m)$ consist of the last entry h, the ascent count a, and these arrays. Define

$$E_{\mathcal{B}}(S) = \{t \in \{0, \dots, a+1\} : m_{t,d,u} = +\infty \text{ for all } (d,u) \in \mathcal{B}\}.$$

For $t \in E_{\mathcal{B}}(S)$, let $T(S,t)$ be the updated state, with $h' = t$ and $a' = a + \mathbf{1}_{h<t}$.

Corollary A.2. *If $c_n(S)$ counts the $\mathcal{B}$-avoiders of length n with state S, then*

$$c_{n+1}(S') = \sum_S c_n(S)\,\big|\{t \in E_{\mathcal{B}}(S) : T(S,t) = S'\}\big|, \qquad A_{\mathcal{B}}(n) = \sum_S c_n(S).$$

The initial condition is $c_1(S(0)) = 1$, with all other values zero, and $A_{\mathcal{B}}(0) = 1$. There are finitely many states at each length. Equal states have the same permitted continuations of every length, with the actual values retained.

Proof. The test for a forbidden pattern and the ascent bound determine the allowed next entries and resulting states. Every nonempty word has a unique prefix and final entry, proving the recurrence by induction. At length n, at most n values i occur, each array indexed by i has size $(D+1)(U+1)$, and every finite coordinate is at most $n-1$, so only finitely many states occur. Equal states permit the same next entries and give equal updated states; induction proves the assertion about continuations. □

The arrays grow with length. For 0120 they record, for each value i, the least larger entry after an occurrence of i and whether two increasing entries larger than i have subsequently appeared. Since the number of states is not fixed, this recurrence is compatible with the failure of P-recursiveness proved in Section 7.

B Finite verification

The supplementary programs compare direct subsequence tests with separate reconstruction and counting routines. The program `verify.py` checks the permutation bounds, reconstruction from monotone and noncrossing subsequences, and the products in the construction by blocks for ascent sequences, weak ascent sequences, and inversion sequences. The program `verify_returns.py` checks the recurrence in Appendix A, the rational formulas and their bijection, containment of the patterns $R_{d,u}$, and the condition on prefixes in Definition 3.1.

The program `verify_arcs.py` compares repetition graph edges with geometric intersections, reconstructs words from retained entries and color arrays alone, tests the coloring argument at small parameters, and verifies (9.1). The program `verify_refinements.py` checks (8.3), the weak ascent condition after moving zeros to the front, the resulting fiber bounds, and (8.2). The program `verify_diagonal.py` checks the decomposition at the first entry at least 2, its coefficient formula, and the recurrence for the number of continuations.

Counting and reconstruction comparisons use integer arithmetic. The archive records commands, bounds, environment, count tables and logs, with numerical optimization diagnostics separated. These finite tests supplement the written proofs; the coloring theorem remains the external input [5].

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